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Intersections of hypergraphs

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Given two weighted k-uniform hypergraphs G, H of order n, how much (or little) can we make them overlap by placing them on the same vertex set? If we place them at random, how concentrated is the distribution of the intersection? The aim of this paper is to investigate these questions.

1. Introduction

The discrepancy of a set of points in a subset of Euclidean space measures how uniformly the points are spread through the set. For instance, the discrepancy of a set of n points in a square of area n can be defined as the maximum difference between the area of a subsquare and the number of points from the set that it contains. Discrepancy theory in the geometric setting has been studied since the work of Weyl [35] on sequences, and is of interest in areas including number theory and combinatorics, as well as having applications in computational geometry and numerical integration (see for instance the books by Beck and Chen [6], Kuipers and Niederreiter [26] and Drmota and Tichy [15]).

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In the discrete context, a similar notion of discrepancy for hypergraphs was introduced by Erdős and Spencer [19], and measures the extent to which the edges of a hypergraph are uniformly distributed (inside the complete graph).[2]

Erdős and Spencer showed that the edges of a k-uniform hypergraph cannot be distributed too uniformly: for every k-uniform hypergraph on n vertices, there is a subset S in which the number of edges differs from $\frac{1}{2}\binom{|S|}{k}$ by at least $c_k n^{(k+1)/2}$. Random hypergraphs show that this bound is optimal up to a constant factor. In the case of graphs (i.e. $k = 2$), Erdős, Goldberg, Pach and Spencer [18] later extended this to graphs of any density p, where the measure of discrepancy is the maximum difference between the number of edges in a subset S and the expected $p\binom{|S|}{2}$. More general results, including an extension of the Erdős–Spencer result to arbitrary densities when $k \geq 3$, were proved in [9].

The aim of this paper is to study the discrepancy of *pairs* of hypergraphs. The discrepancy of a pair of hypergraphs, introduced in [10], measures the extent to which the edges of the two hypergraphs are uniformly and independently distributed. Given k-uniform hypergraphs G and H with n vertices and densities p, q, the discrepancy of the pair G, H is the maximum size, over all bijections between their vertex sets, of the difference between their intersection and $pq\binom{n}{k}$ (the expected intersection under a random mapping). For instance, G and H have discrepancy 0 if their intersection has the same size for any placement of both hypergraphs onto the same vertex set; on the other hand, if G and H are isomorphic to the same incomplete graph then their discrepancy will be large, as any isomorphism between them will give a much larger than average intersection.

In light of the results of Erdős and Spencer [19], it is natural to expect that every pair of (unweighted) k-uniform hypergraphs of moderate density should have large discrepancy (of order at least $n^{(k+1)/2}$), and we conjectured in [10] that this should be the case. For $k = 2$, this conjecture was proved in [10], but for $k = 3$ we will show here that there is a counterexample (see Section 1.2); for $k \geq 4$, the conjecture is still open.

In this paper, we will mainly be interested in the discrepancy of pairs of *weighted* hypergraphs. It turns out that, for weighted hypergraphs the picture is dramatically different from the unweighted case:

- For every $k \geq 1$, there is a set of k nontrivial weighted k-uniform hypergraphs such that every pair has discrepancy 0.

On the other hand, if we take one additional hypergraph, there must be a pair with large discrepancy:

- For every $k \geq 1$, and every set of $k + 1$ nontrivial normalized weighted hypergraphs, there is some pair that has discrepancy at least $c_k n^{(k+1)/2}$.

[2] There are a number of other standard ways to measure discrepancy for discrete structures: see Beck and Sós [7], Sós [32], Chazelle [12] and Matoušek [29].