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On the ultimate categorical independence ratio

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ABSTRACT

Brown, Nowakowski and Rall defined the ultimate categorical independence ratio of a graph G as $A(G) = \lim_{k\to\infty} i(G^{\times k})$, where $i(G) = \frac{\alpha(G)}{|V(G)|}$ denotes the independence ratio of a graph G, and $G^{\times k}$ is the kth categorical power of G. Let $a(G) = \max\{\frac{|U|}{|U|+|N_G(U)|}\colon U$ is an independent set of $G\}$, where $N_G(U)$ is the neighborhood of U in G. In this paper we answer a question of Alon and Lubetzky, namely we prove that $A(G) = a(G)$ if $a(G) \leqslant \frac{1}{2}$, and $A(G) = 1$ otherwise. We also discuss some other open problems related to $A(G)$ which are immediately settled by this result.

1. Introduction

The *independence ratio* of a graph G is defined as $i(G) = \frac{\alpha(G)}{|V(G)|}$, that is, as the ratio of the independence number and the number of vertices. For two graphs G and H, their *categorical product* (also called as direct or tensor product) $G \times H$ is defined on the vertex set $V(G \times H) = V(G) \times V(H)$ with edge set

$$E(G \times H) = \big\{\{(x_1, y_1), (x_2, y_2)\}\colon \{x_1, x_2\} \in E(G) \text{ and } \{y_1, y_2\} \in E(H)\big\}.$$

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The kth categorical power $G^{\times k}$ is the k-fold categorical product of G. The *ultimate categorical independence ratio* of a graph G is defined as

$$A(G) = \lim_{k\to\infty} i\bigl(G^{\times k}\bigr).$$

This parameter was introduced by Brown, Nowakowski and Rall in [2] where they proved that for any independent set U of G the inequality $A(G) \geqslant \frac{|U|}{|U|+|N_G(U)|}$ holds, where $N_G(U)$ denotes the neighborhood of U in G. Furthermore, they showed that $A(G) > \frac{1}{2}$ implies $A(G) = 1$.

Motivated by these results, Alon and Lubetzky [1] defined the parameters $a(G)$ and $a^*(G)$ as follows

$$a(G) = \max_{U \text{ is independent set of } G} \frac{|U|}{|U| + |N_G(U)|} \quad \text{and} \quad a^*(G) = \begin{cases} a(G) & \text{if } a(G) \leqslant \frac{1}{2}, \\ 1 & \text{if } a(G) > \frac{1}{2}, \end{cases}$$

and they proposed the following two questions.

Question 1. (See [1].) Does every graph G satisfy $A(G) = a^*(G)$? Or, equivalently, does every graph G satisfy $a^*(G^{\times 2}) = a^*(G)$?

Question 2. (See [1].) Does the inequality $i(G \times H) \leqslant \max\{a^*(G), a^*(H)\}$ hold for every two graphs G and H?

The above results from [2] give us the inequality $A(G) \geqslant a^*(G)$. One can easily see the equivalence between the two forms of Question 1; moreover, it is not hard to show that an affirmative answer to Question 1 would imply the same for Question 2 (see [1]).

Following [2] a graph G is called *self-universal* if $A(G) = i(G)$. As a consequence, the equality $A(G) = a^*(G)$ in Question 1 is also satisfied for these graphs according to the chain inequality $i(G) \leqslant a(G) \leqslant a^*(G) \leqslant A(G)$. Regular bipartite graphs, cliques and Cayley graphs of Abelian groups belong to this class (see [2]). In [3] the author proved that a complete multipartite graph G is self-universal, except for the case when $i(G) > \frac{1}{2}$. Therefore the equality $A(G) = a^*(G)$ is also verified for this class of graphs. (In the latter case $A(G) = a^*(G) = 1$.) In [1] it is shown that the graphs which are disjoint union of cycles and complete graphs satisfy the inequality in Question 2.

In this paper we answer Question 1 affirmatively, thereby also obtaining a positive answer for Question 2. Moreover it solves some other open problems related to $A(G)$. In the proofs we exploit an idea of Zhu [4] that he used on the way when proving the fractional version of Hedetniemi's conjecture. In Section 2 this tool is presented. Then, in Section 3, first we prove the inequality

$$i(G \times H) \leqslant \max\bigl\{a(G), a(H)\bigr\}, \quad \text{for every two graphs } G \text{ and } H,$$