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Ore's conjecture on color-critical graphs is almost true

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ABSTRACT

A graph G is *k-critical* if it has chromatic number k, but every proper subgraph of G is $(k-1)$-colorable. Let $f_k(n)$ denote the minimum number of edges in an n-vertex k-critical graph. We give a lower bound, $f_k(n) \geq F(k,n)$, that is sharp for every $n = 1 \pmod{k-1}$. The bound is also sharp for $k = 4$ and every $n \geq 6$. The result improves a bound by Gallai and subsequent bounds by Krivelevich and Kostochka and Stiebitz, and settles the corresponding conjecture by Gallai from 1963. It establishes the asymptotics of $f_k(n)$ for every fixed k. It also proves that the conjecture by Ore from 1967 that for every $k \geq 4$ and $n \geq k+2$, $f_k(n+k-1) = f_k(n) + \frac{k-1}{2}(k - \frac{2}{k-1})$ holds for each $k \geq 4$ for all but at most $k^3/12$ values of n. We give a polynomial-time algorithm for $(k-1)$-coloring of a graph G that satisfies $|E(G[W])| < F(k,|W|)$ for all $W \subseteq V(G)$, $|W| \geq k$. We also present some applications of the result.

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1. Introduction

A *proper k-coloring*, or simply *k-coloring*, of a graph $G = (V, E)$ is a function $f : V \to \{1, 2, \dots, k\}$ such that for each $uv \in E$, $f(u) \neq f(v)$. A graph G is *k-colorable* if there exists a k-coloring of G. The *chromatic number*, $\chi(G)$, of a graph G is the smallest k such that G is k-colorable. A graph G is *k-chromatic* if $\chi(G) = k$.

A graph G is *k-critical* if G is not $(k-1)$-colorable, but every proper subgraph of G is $(k-1)$-colorable. Then every k-critical graph has chromatic number k and every k-chromatic graph contains a k-critical subgraph. This means that some problems for k-chromatic graphs may be reduced to problems for k-critical graphs, whose structure is more restricted. For example, every k-critical graph is 2-connected and $(k-1)$-edge-connected. Critical graphs were first defined and used by Dirac [4–6] in 1951–1952.

The only 1-critical graph is K_1, and the only 2-critical graph is K_2. The only 3-critical graphs are the odd cycles. For every $k \geq 4$ and every $n \geq k+2$, there exists a k-critical n-vertex graph. Let $f_k(n)$ be the minimum number of edges in a k-critical graph with n vertices. Since $\delta(G) \geq k-1$ for every k-critical n-vertex graph G,

$$f_k(n) \geq \frac{k-1}{2}n \tag{1}$$

for all $n \geq k$, $n \neq k+1$. Equality is achieved for $n = k$ and for $k = 3$ and n odd. Brooks' Theorem [3] implies that for $k \geq 4$ and $n \geq k+2$, the inequality in (1) is strict. In 1957, Dirac [8] asked to determine $f_k(n)$ and proved that for $k \geq 4$ and $n \geq k+2$,

$$f_k(n) \geq \frac{k-1}{2}n + \frac{k-3}{2}. \tag{2}$$

The result is tight for $n = 2k-1$ and yields $f_k(2k-1) = k^2 - k - 1$. Dirac used his bound to evaluate chromatic number of graphs embedded into fixed surfaces. Later, Kostochka and Stiebitz [17] improved (2) to

$$f_k(n) \geq \frac{k-1}{2}n + k - 3 \tag{3}$$

when $n \neq 2k-1, k$. This yields $f_k(2k) = k^2 - 3$ and $f_k(3k-2) = \frac{3k(k-1)}{2} - 2$. In his fundamental papers [10,11], Gallai found exact values of $f_k(n)$ for $k+2 \leq n \leq 2k-1$:

Theorem 1. *(See Gallai [11].) If $k \geq 4$ and $k+2 \leq n \leq 2k-1$, then*

$$f_k(n) = \frac{1}{2}\big((k-1)n + (n-k)(2k-n)\big) - 1.$$

He also proved the following general bound for $k \geq 4$ and $n \geq k+2$:

$$f_k(n) \geq \frac{k-1}{2}n + \frac{k-3}{2(k^2-3)}n. \tag{4}$$