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Colouring edges with many colours in cycles

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A R T I C L E I N F O

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A B S T R A C T

The arboricity of a graph G is the minimum number of colours needed to colour the edges of G so that every cycle gets at least two colours. Given a positive integer p, we define the generalized p-arboricity $\mathrm{Arb}_p(G)$ of a graph G as the minimum number of colours needed to colour the edges of a multigraph G in such a way that every cycle C gets at least $\min(|C|, p+1)$ colours. In the particular case where G has girth at least $p+1$, $\mathrm{Arb}_p(G)$ is the minimum size of a partition of the edge set of G such that the union of any p parts induces a forest. In this paper, we relate the generalized p-arboricity of a graph G to the maximum density of a multigraphs having a shallow subdivision (where edges are becoming paths of length at most p) as a subgraph of G, by proving that each of these values is bounded by a polynomial function of the other.

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1. Introduction

In this paper, we consider the following problem: given a graph G, how many colours do we need to colour the edges of G in such a way that every cycle gets "many" colours? Of course, the answer to this question depends on the precise meaning of "many". If we require that each cycle γ of length l of G gets l colours, i.e., every cycle is a rainbow, then the minimum number of colours needed is equal to the maximum size of a block of G, as two edges of G belong to a common cycle if and only if they belong to the same block. If we require that every cycle gets at least 2 colours, i.e., every colour class induces a forest, then the minimum number of colours needed is the *arboricity* $\mathrm{Arb}(G)$ of G, and its determination is solved by the well-known Nash-Williams' theorem we recall now.

Denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G. Also denote by $|G| = |V(G)|$ (resp. $\|G\| = |E(G)|$) the *order* of G (resp. *size*). For $A \subseteq V(G)$ denote by $G[A]$ the subgraph of G induced by A. By Nash-Williams' theorem [9,10], the arboricity of a graph G is given by the formula:

$$\mathrm{Arb}(G) = \max_{A \subseteq V(G),\, |A|>1} \left\lceil \frac{\|G[A]\|}{|A|-1} \right\rceil. \tag{1}$$

Here we consider a generalization of these two extreme cases. A general form of our problem is captured by the following:

> Given an unbounded non-decreasing function $f : \mathbb{N} \to \mathbb{N}$ and an integer p, what is the minimum number $N_f(G, p)$ of colours needed to colour the edges of a graph G in such a way that each cycle γ gets at least $\min(f(|\gamma|), p + 1)$ colours?

Thus for $p = 1$ and $f(n) \geq 2$ we get $N_f(G, p) = \mathrm{Arb}(G)$. For an arbitrary graph G, it is usually difficult to determine $N_f(G, p)$. Our interest is to find upper bound for $N_f(G, p)$ in terms of other graph parameters, and upper bound for $N_f(G, p)$ for some nice classes of graphs and/or for some nice special functions f.

Many colouring parameters are bounded for proper minor closed classes of graphs. It is natural to ask for which functions f is $N_f(G, p)$ bounded for any proper minor closed class $\mathcal{C}$ of graphs. We shall prove (Lemma 1) that if $f(2^{p-1}) > p - 1$ for some value of p then there is a (quite small) minor closed class of graphs $\mathcal{C}$, such that $N_f(G, p)$ is unbounded. On the other hand, we prove (Corollary 6) that if $f(x) \leq \lceil \log_2 x \rceil$ for all x then $N_f(G, p)$ is not only bounded on proper minor closed classes of graphs, but actually bounded on a class $\mathcal{C}$ if and only if $\mathcal{C}$ has *bounded expansion* (to be defined in Section 3).

Next we consider the special function $f(x) = x$. For this special function, the parameter $N_f(G, p + 1)$ is denoted as $\mathrm{Arb}_p(G)$ and is called the *generalized p-arboricity* of G. So $\mathrm{Arb}_p(G)$ is the number of colours needed if we require that each cycle of G gets at least $p + 1$ colours or is a rainbow if its length is smaller than $p + 1$. Note that if $p = 1$, then $\mathrm{Arb}_p(G)$ is the arboricity $\mathrm{Arb}(G)$ of G. We shall relate the generalized p-arboricities