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The inducibility of blow-up graphs ☆

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ABSTRACT

The blow-up of a graph is obtained by replacing every vertex with a finite collection of copies so that the copies of two vertices are adjacent if and only if the originals are. If every vertex is replaced with the same number of copies, then the resulting graph is called a balanced blow-up.

We show that any graph which contains the maximum number of induced copies of a sufficiently large balanced blow-up of H is itself essentially a blow-up of H. This gives an asymptotic answer to a question in [2].

1. Introduction

What is the maximum number of induced copies of a given graph H in any graph on n vertices? This basic question has been studied previously in [2,3,6,11,4,9], and its answer is known for a handful of small graphs and certain complete multipartite graphs. Obtaining a general answer seems to be difficult. Even the case of the path on four vertices is not resolved. In this paper we provide an asymptotic answer to this question for the class of graphs described below.

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The blow-ups of H provide natural candidates for graphs that contain many induced copies of H. For a positive integer vector $\mathbf{k} \in \mathbb{Z}_+^{V(G)}$, the $\mathbf{k}$-*blow-up* of H, denoted by $H^{(\mathbf{k})}$, is the graph obtained by replacing every vertex v of H with $\mathbf{k}(v)$ different vertices where a copy of u is adjacent to a copy of v in the blow-up graph if and only if u is adjacent to v in H. When all $\mathbf{k}(v)$ are equal to some positive integer k, the corresponding blow-up is called *balanced* and denoted simply by $H^{(k)}$.

For two graphs H and G, the *induced density* of H in G, denoted by $i(H,G)$ is the number of induced copies of H in G divided by $\binom{|V(G)|}{|V(H)|}$. Let $i(H,n)$ denote the maximum induced density of H in any graph on n vertices. Modern extremal graph theory more often focuses on understanding the asymptotic behavior of such functions rather than determining their values on every given n. The asymptotic behavior of $i(H,n)$ is captured by the *inducibility* of H defined as $i(H) := \lim_{n\to\infty} i(H,n)$. It is straightforward to see that this limit always exists.

The blow-ups of the graph H provide natural candidates for graphs with largest induced density of H. Note that graphs with maximum edge density are always complete, but the blow-ups of a one-edge graph are complete bipartite graphs. As this simple example already illustrates, it is not in general true that the maximal graphs are always the blow-ups of H.

In contrast with the above example, our main result, Theorem 3.2, says that for every graph H, the inducibility of $H^{(h)}$ is essentially achieved by blow-ups of H. More precisely, for sufficiently large h, there exist vectors $\mathbf{k}_n \in \mathbb{Z}_+^{V(H)}$ with $|V(H^{(\mathbf{k}_n)})| = n$ such that

$$i\big(H^{(h)}\big) = \lim_{n\to\infty} i\big(H, H^{(\mathbf{k}_n)}\big).$$

Note that $K_r^{(h)}$, the h-blow-up of the complete graph on r vertices, is the complete r-partite graph where each part has exactly h vertices. Bollobás, Egawa, Harris and Jin [2] showed that for sufficiently large h, the maximal graphs for $K_r^{(h)}$ are blow-ups of K_r. They asked for which graphs H does there exist a constant h such that for sufficiently large n, the graph on n vertices that contains the maximum number of induced copies of $H^{(h)}$ is a blow-up of H? Note that Theorem 3.2 gives an asymptotic answer to this question.

2. Preliminaries

All graphs in this paper are finite and loopless. We denote vectors with bold font, e.g. $\mathbf{a} = (\mathbf{a}(1), \mathbf{a}(2), \mathbf{a}(3))$ is a vector with three coordinates. For every positive integer k, let $[k]$ denote the set $\{1, \dots, k\}$. The *adjacency matrix* of a graph G, denoted by A_G, is a zero–one matrix whose rows and columns are indexed by vertices of G and $A_G(u,v) = 1$ if and only if u is adjacent to v. For a graph G and a subset $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$.

A *homomorphism* from a graph H to a graph G is a mapping $\phi : V(H) \to V(G)$ such that if uv is an edge of H, then $\phi(u)\phi(v)$ is an edge of G. A *strong homomorphism* from