Journal of Combinatorial Theory, Series B 109 (2014) 228–249

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Tournaments with near-linear transitive subsets

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ARTICLE INFO

Article history:
Received 9 July 2012
Available online 8 July 2014

Keywords:
The Erdős–Hajnal conjecture
Tournaments

ABSTRACT

Let H be a tournament, and let $\epsilon \geq 0$ be a real number. We call ϵ an "Erdős–Hajnal coefficient" for H if there exists $c > 0$ such that in every tournament G not containing H as a subtournament, there is a transitive subset of cardinality at least $c|V(G)|^{\epsilon}$. The Erdős–Hajnal conjecture asserts, in one form, that every tournament H has a positive Erdős–Hajnal coefficient. This remains open, but recently the tournaments with Erdős–Hajnal coefficient 1 were completely characterized. In this paper we provide an analogous theorem for tournaments that have an Erdős–Hajnal coefficient larger than 5/6; we give a construction for them all, and we prove that for any such tournament H there are numbers c, d such that, if a tournament G with $|V(G)| > 1$ does not contain H as a subtournament, then $V(G)$ can be partitioned into at most $c(\log(|V(G)|))^d$ transitive subsets.

1. Introduction

A *tournament* is a loopless digraph such that for every pair of distinct vertices u, v, exactly one of uv, vu is an edge. A *transitive set* is a subset of $V(G)$ that can be ordered

[1] Supported by NSF grant IIS-1117631.
[2] Supported by NSF grants DMS-1001091 and IIS-1117631.
[3] Supported by ONR grant N00014-10-1-0680 and NSF grant DMS-0901075.

http://dx.doi.org/10.1016/j.jctb.2014.06.007
0095-8956/

$\{x_1, \ldots, x_k\}$ such that $x_i x_j$ is an edge for $1 \le i < j \le k$. A *colouring* of a tournament G is a partition of $V(G)$ into transitive sets, and the *chromatic number* $\chi(G)$ is the minimum number of transitive sets in a colouring. If G, H are tournaments, we say that G is *H-free* if no subtournament of G is isomorphic to H.

There are some tournaments H with the property that every H-free tournament has chromatic number at most a constant (depending on H). These are called *heroes*, and they were all explicitly found in an earlier paper [3]. In this paper, we turn to the question: which are the most heroic non-heroes? It turns out that for some non-heroes H, the chromatic number of every H-free tournament G is at most a polylog function of the number of vertices of G, and all the others give nothing better than a polynomial bound. More exactly, we will show the following (we will often write $|G|$ instead of $|V(G)|$, when G is a graph or tournament):

1.1. *Every tournament H has exactly one of the following properties:*

- *for some c, every H-free tournament has chromatic number at most c (that is, H is a hero)*
- *for some c, d, every H-free tournament G with $|G| > 1$ has chromatic number at most $c(\log(|G|))^d$, and for all c, there are H-free tournaments G with $|G| > 1$ and with chromatic number at least $c(\log(|G|))^{1/3}$*
- *for all c, there are H-free tournaments G with $|G| > 1$ and with chromatic number at least $c|G|^{1/6}$.*

This is one of our main results. The other is an explicit construction for all tournaments of the second type, which we call *pseudo-heroes*.

This research is closely connected with, and motivated by, the Erdős–Hajnal conjecture. P. Erdős and A. Hajnal [7] made the following conjecture in 1989 (it is still open):

1.2 *(The Erdős–Hajnal conjecture). For every graph H there exists a number $\epsilon > 0$ such that every graph G that does not contain H as an induced subgraph contains a clique or a stable set of size at least $|G|^\epsilon$.*

If G is a tournament, $\alpha(G)$ denotes the cardinality of the largest transitive subset of $V(G)$. It was shown in [1] that Conjecture 1.2 is equivalent to the following:

1.3 *(Conjecture). For every tournament H there exists a number $\epsilon > 0$ such that every H-free tournament G satisfies $\alpha(G) \ge |G|^\epsilon$.*

Let us say that $\epsilon \ge 0$ is an *EH-coefficient* for a tournament H if there exists $c > 0$ such that every H-free tournament G satisfies $\alpha(G) \ge c|G|^\epsilon$. Thus, the Erdős–Hajnal conjecture is equivalent to the conjecture that every tournament has a positive EH-coefficient.