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A characterization of a family of edge-transitive metacirculant graphs ☆

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ABSTRACT

A characterization is given of the class of edge-transitive Cayley graphs of Frobenius groups $\mathbb{Z}_{r^d}{:}\mathbb{Z}_m$ with r an odd prime and m odd, of valency less than $2p_1$ with p_1 the smallest prime divisor of m. It is shown that either $(r^d, m) = (p, \frac{p-1}{2})$ or $(29, 7)$, or such a graph is a normal Cayley graph and half-transitive. This provides new construction of half-transitive graphs.

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1. Introduction

By $\Gamma = (V, E)$ we mean a graph with vertex set V and edge set E. A graph Γ is called *X-vertex-transitive* or *X-edge-transitive* if $X \leqslant \mathsf{Aut}\,\Gamma$ is transitive on V or on E, respectively. A *circulant* is a graph Γ such that $\mathsf{Aut}\,\Gamma$ contains a cyclic subgroup which is transitive on V. A graph Γ is a *metacirculant* if $\mathsf{Aut}\,\Gamma$ contains a metacyclic subgroup that is transitive on V.

Edge-transitive circulants have been characterized by Kovács [11] and Li [15] independently. It would be a natural next step towards a characterization of edge-transitive metacirculants. Some special cases have been done in the literature: see [5] for the case of order a product of two primes; see [17] for the case of prime-power order; see [12] for the study of arc-regular dihedrants. In this paper, we characterize edge-transitive metacirculants that admit a vertex-transitive Frobenius subgroup $\mathbb{Z}_{r^d}{:}\mathbb{Z}_m$.

An *arc* of a graph Γ is an ordered pair of adjacent vertices. A graph Γ is *arc-transitive* if $\mathsf{Aut}\,\Gamma$ is transitive on the set of arcs of Γ. Obviously, an arc-transitive graph is edge-transitive. A graph $\Gamma = (V, E)$ is called *half-transitive* if $\mathsf{Aut}\,\Gamma$ is transitive on both V and E, and intransitive on the arc set. Constructing and characterizing half-transitive graphs has received considerable attention in algebraic graph theory, see [1,18,19,22] for references. Edge-transitive graphs of valency at most 6 and restricted order have been characterized for various special cases, refer to [6–8,13,24]. It will be shown that most edge-transitive metacirculants of small valency that admits a vertex-transitive Frobenius automorphism subgroup $\mathbb{Z}_{r^d}{:}\mathbb{Z}_m$ are half-transitive, see Corollary 1.2.

A graph Γ is a *Cayley graph* if there exists a group G and a subset $1 \notin S \subset G$ with $S = S^{-1} = \{s^{-1} \mid s \in S\}$ such that the vertex set V can be identified with G and x is adjacent to y if and only if $yx^{-1} \in S$. This Cayley graph is denoted by $\mathsf{Cay}(G, S)$. It is known that a graph Γ is a Cayley graph of a group G if and only if $\mathsf{Aut}\,\Gamma$ contains a subgroup that is isomorphic to G and regular on the vertex set, see [2, Lemma 16.3]. Further, if $\mathsf{Aut}\,\Gamma$ contains a normal regular subgroup G then Γ is called a *normal Cayley graph* of G. The main result of this paper is the following theorem.

Theorem 1.1. *Let $G = \langle a\rangle{:}\langle b\rangle \cong \mathbb{Z}_{r^d}{:}\mathbb{Z}_m$ be a Frobenius group, where r is an odd prime and m is an odd integer. Let Γ be a connected X-edge-transitive graph of valency $\mathsf{val}\,\Gamma < 2p_1$, where p_1 is the smallest prime divisor of m, and X contains a vertex transitive subgroup isomorphic to G. Then one of the following statements holds*:

(i) $\mathsf{Aut}\,\Gamma = X = G{:}X_\alpha = G{:}\mathbb{Z}_k$ *is soluble, $k \mid (r-1)$, there exists $\sigma \in \mathsf{Aut}(G)$ of order k such that Γ is isomorphic to one of the following normal Cayley graphs*:

$$\mathsf{Cay}(G, S_j), \quad \textit{where } S_j = \{b^j, b^{m-j}\}^{\langle\sigma\rangle},\ (j, m) = 1,\ \textit{and } 1 \leqslant j \leqslant m/2,$$

and Γ is half-transitive;

(ii) $G = \mathbb{Z}_{11}{:}\mathbb{Z}_5$, $X = \mathbb{Z}_{11}{:}\mathbb{Z}_{10}$, $\mathsf{Aut}\,\Gamma = \mathrm{PGL}(2, 11)$, *and Γ is arc-transitive of valency* 4;

(iii) *X is almost simple, and the triple (X, G, X_α) lies in Table 1.*