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A base exchange property for regular matroids

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ABSTRACT

In this paper, we show that for any two bases B and B' of a regular matroid, there is an element $e \in B$ such that there is a unique element $f \in B'$ for which both $(B\backslash\{e\}) \cup \{f\}$ and $(B'\backslash\{f\}) \cup \{e\}$ are bases of M. This solves a problem posed by White in 1980.

1. Introduction

Let M be a matroid and let B and B' be bases of M. We say that the triple (M, B, B') has the **unique exchange property** (UE) if there exists an element $e \in B$ for which there is a unique element $f \in B'$ such that both $(B\backslash\{e\}) \cup \{f\}$ and $(B'\backslash\{f\}) \cup \{e\}$ are bases of M. We say that these bases are obtained from B and B' by a *unique exchange*. In this paper, we resolve a problem of White from 1980 in the affirmative by showing the following:

Theorem 1.1. *For any regular matroid M and any pair of bases B, B', the triple (M, B, B') has the unique exchange property.*

White's problem is also listed as Problem 14.8.11 in Oxley's list [2]. Note that the answer to the above problem is negative for binary matroids in general. To see this, take

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0095-8956/

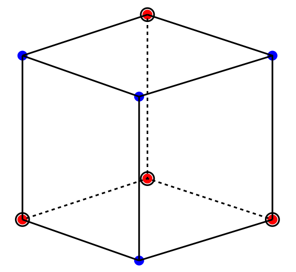

Fig. 1. Two bases in $AG(2,3)$ not having the property (UE).

two disjoint bases in the affine geometry $AG(3,2)$ as shown in Fig. 1. Then it is seen that these bases do not have the unique exchange property.

Until now, little progress had been made on this problem, although related work in [1] and [8] can be found. To give a bit of background on the origins of the problem, we give the following definitions introduced in [7]. Let $\mathcal{B} = (B_1, B_2, \ldots, B_k)$ be a sequence of bases of a matroid M. For another sequence of k bases $\mathcal{B}' = (B'_1, B'_2, \ldots, B'_k)$, we write $\mathcal{B} \simeq \mathcal{B}'$ whenever the sequence $\mathcal{B}'$ may be obtained from $\mathcal{B}$ by a composition of unique exchanges and permutations of the bases. We say that the sequences $\mathcal{B}$ and $\mathcal{B}'$ are **compatible** if for each $e \in M$, the sets $\{B_i \in \mathcal{B} \mid e \in B_i\}$ and $\{B'_i \in \mathcal{B}' \mid e \in B'_i\}$ have the same cardinality. Clearly if $\mathcal{B} \simeq \mathcal{B}'$, then $\mathcal{B}$ and $\mathcal{B}'$ must be compatible. In [7, Conjecture 8], it was conjectured that if $\mathcal{B}$ and $\mathcal{B}'$ are compatible sequences of bases in a regular matroid, then $\mathcal{B} \simeq \mathcal{B}'$. The motivation behind this conjecture comes from the study of the bracket ring of a matroid (see [5] and [6]). Very little progress has been made on this conjecture, and indeed, just showing that any pair of bases in a regular matroid has the unique exchange property is hard. This was mentioned as an open problem in [7].

To prove Theorem 1.1, we shall use the minimum counterexample approach. That is, we shall assume that the theorem is false, and let (M, R, B) be a triple not having the property (UE) where M is regular, R and B are bases, and $|E(M)|$ is minimum among all such triples. An easy proof shows that M cannot be graphic or cographic. In Section 2.1, we show that M must be 3-connected. From this point onwards, the proof of the above theorem relies heavily on the structure of regular matroids given by Seymour's decomposition theorem [3] which implies that 3-connected regular matroids other than R_{10} can be built up via 3-sums of graphic or cographic matroids. Among other things, this implies that M can be written as a 3-sum $M = M_1 \oplus M_2$ where M_1 is either graphic or cographic. The basic strategy of the proof is to first determine what the matroid M_1 looks like in the graphic case. Much of the proof is spent doing this (in Section 5). It turns out that $M_1 \simeq M(K_5 \backslash e)$. This can then be used to show that $M_1 \simeq M^*(K_{3,3})$ in the cographic case.

In Sections 7 and 8, we complete the proof of the theorem. Here we exploit the fact that the "leaves" of M (defined in Section 7) have a specific structure. Of particular