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Rao's degree sequence conjecture

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ABSTRACT

Let us say two (simple) graphs G, G' are *degree-equivalent* if they have the same vertex set, and for every vertex, its degrees in G and in G' are equal. In the early 1980's, S.B. Rao made the conjecture that in any infinite set of graphs, there exist two of them, say G and H, such that H is isomorphic to an induced subgraph of some graph that is degree-equivalent to G. We prove this conjecture.

1. Introduction

Neil Robertson and the second author proved in [7] that the class of all graphs forms a "well-quasi-order" under minor containment, that is, that in every infinite set of graphs, one of its members is a minor of another. The same is not true for induced subgraph containment, but a conjecture of S.B. Rao [6] proposed a way to tweak the latter containment relation to make it a well-quasi-order; and in this paper we prove Rao's conjecture.

Let us be more precise. All graphs and digraphs in this paper are finite and without loops or parallel edges, and digraphs do not have directed cycles of length two. If G is a graph and $X \subseteq V(G)$, we denote by $G|X$ the subgraph of G induced on X (that is, the subgraph with vertex set X and edge set all edges of G with both ends in X); and we say that $G|X$ is an *induced subgraph* of G. Let us say two graphs G, G' are *degree-equivalent* if

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they have the same vertex set, and for every vertex, its degrees in G and in G' are equal; and H is *Rao-contained* in G if H is isomorphic to an induced subgraph of some graph that is degree-equivalent to G. In the early 1980's, S.B. Rao [6] made the conjecture, the main theorem of this paper, that:

1.1. *In any infinite set of graphs, there exist two of them, say G and H, such that H is Rao-contained in G.*

A *quasi-order* Q consists of a class $E(Q)$ and a transitive reflexive relation which we denote by $\leqslant$ or $\leqslant_Q$; and it is a *well-quasi-order* or *wqo* if for every infinite sequence q_i ($i = 1, 2, \ldots$) of elements of $E(Q)$ there exist $j > i \geqslant 1$ such that $q_i \leqslant_Q q_j$. Rao-containment is transitive (this is an easy exercise), and so the following is a reformulation of 1.1:

1.2. *The class of all graphs, ordered by Rao-containment, is a wqo.*

The proof falls into three main parts, and let us sketch them here. A "split graph" is a graph such that there is a partition of its vertex set into a stable set and a clique. For Rao-containment of split graphs, we will require the vertex set injection to preserve this partition. A "k-rooted graph" means (roughly) a graph with k of its vertices designated as roots. For Rao-containment of k-rooted graphs, we require the vertex set injection to respect the roots. (This will all be said more precisely later.) We show three things:

- For every graph H, if G is a graph that does not Rao-contain H, then $V(G)$ can be partitioned into two sets (except for a bounded number of vertices), the first inducing a split graph and the second inducing a graph of bounded degree (or the complement of one), such that the edges between these two sets are under control. This allows us to break G into two parts; but both parts acquire a bounded number of roots, because we need to remember how to hook them back together to form G. This is proved in 4.2.
- For all k, the k-rooted graphs of bounded degree (except for the roots) form a wqo under Rao-containment. This is proved in 6.1.
- For all k, the k-rooted split graphs also form a wqo under Rao-containment. This is proved in 7.2.

From these three statements, the truth of 1.2 follows in a few lines, and is given immediately after 7.2. Then the proof of 7.2 occupies the remainder of the paper.

2. Rao-containment in fixed position

We need to study the structure of the graphs that do not Rao-contain a fixed graph H. For there to be a Rao-containment of H in G, there must be an injection of $V(H)$ into $V(G)$, and a graph G' degree-equivalent to G, such that the injection is an isomorphism