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On spanning tree packings of highly edge connected graphs

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ABSTRACT

We prove a refinement of the tree packing theorem by Tutte/Nash-Williams for finite graphs. This result is used to obtain a similar result for end faithful spanning tree packings in certain infinite graphs and consequently to establish a sufficient Hamiltonicity condition for the line graphs of such graphs.

1. Introduction

A spanning tree packing of a graph is a set of edge disjoint spanning trees. The following theorem, discovered independently by Tutte and Nash-Williams, provides a sufficient condition for the existence of a spanning tree packing of cardinality k.

Theorem 1.1. *(See Tutte [28], Nash-Williams [25].) Every finite $2k$-edge connected graph has k edge disjoint spanning trees.*

It is known that this result does not remain true for infinite graphs, not even for locally finite graphs, i.e., infinite graphs where every vertex has only finitely many neighbours. Aharoni and Thomassen [1] showed that it is possible to construct locally finite graphs of arbitrarily high edge connectivity that do not possess two edge disjoint spanning trees.

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Meanwhile the following approach due to Diestel and Kühn [15,16] allows a natural extension of the result to locally finite graphs. They proposed to use topological notions of paths and cycles in infinite graphs, which has the advantage of being able to define cycles containing infinitely many edges. More precisely they used homeomorphic images of the unit interval and the unit circle in the Freudenthal compactification of a graph (so called arcs and topological circles) as infinite analogues of paths and cycles in finite graphs. They also introduced the concept of a topological spanning tree, which is an infinite analogue of finite spanning trees compatible with the notions of arcs and topological circles.

Using the topological notions above numerous results from finite graph theory have been generalised to locally finite graphs [2–8,12,13,15–19,21,26], and even to general topological spaces [20,29], substantiating their impact on infinite graph theory. As mentioned earlier these new concepts can also be used to establish a generalisation of Theorem 1.1. The following result can be found in [14].

Theorem 1.2. *Every locally finite $2k$-edge connected graph has k edge disjoint topological spanning trees.*

The starting point of this paper was the following conjecture by Georgakopoulos related to the Hamiltonian problem in infinite graphs.

Conjecture 1.3. *(See Georgakopoulos [17].) The line graph $L(G)$ of every locally finite 4-edge connected graph G has a Hamiltonian circle.*

By an infinite Hamiltonian circle we mean a topological circle containing every vertex and every end. In the finite case Conjecture 1.3 is known to be true. This was first observed by Thomassen [27] who stated the fact without proof. A simple proof later given by Catlin [9] makes use of Theorem 1.1. However, it turns out that Theorem 1.2 is insufficient to provide a similar proof for Conjecture 1.3. Hence we need a generalisation of the theorem of Tutte/Nash-Williams involving a better notion of spanning trees.

In the present paper we establish a sufficient condition for the existence of large end faithful spanning tree packings similar to Theorem 1.1 where an end faithful spanning tree packing is a spanning tree packing in which every tree is end faithful.

Theorem 1.4. *Let G be a $2k$-edge connected locally finite graph with at most countably many ends. Then G admits an end faithful spanning tree packing of cardinality $k-1$.*

We also show that for a graph with at most countably many ends the topological spanning tree packing in Theorem 1.2 can be chosen in a way that the union of any two of the topological spanning trees is a connected end faithful subgraph of G.

Theorem 1.5. *Let G be a $2k$-edge connected locally finite graph with at most countably many ends. Then G admits a topological spanning tree packing $\mathcal{T}$ of cardinality k such*