Journal of Combinatorial Theory, Series B 104 (2014) 28–59

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

3-choosability of planar graphs with (⩽4)-cycles far apart

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ARTICLE INFO

Article history:
Received 22 January 2011
Available online 11 November 2013

Keywords:
Planar graphs
List coloring

ABSTRACT

A graph is k-choosable if it can be colored whenever every vertex has a list of at least k available colors. We prove that if cycles of length at most four in a planar graph G are pairwise far apart, then G is 3-choosable. This is analogous to the problem of Havel regarding 3-colorability of planar graphs with triangles far apart.

1. Introduction

All graphs considered in this paper are simple and finite. The concepts of list coloring and choosability were introduced by Vizing [13] and independently by Erdős et al. [7]. A *list assignment* of G is a function L that assigns to each vertex $v \in V(G)$ a list $L(v)$ of available colors. An L*-coloring* is a function $\varphi : V(G) \to \bigcup_v L(v)$ such that $\varphi(v) \in L(v)$ for every $v \in V(G)$ and $\varphi(u) \neq \varphi(v)$ whenever u and v are adjacent vertices of G. If G admits an L-coloring, then it is L*-colorable*. A graph G is k*-choosable* if it is L-colorable for every list assignment L such that $|L(v)| \geqslant k$ for all $v \in V(G)$. The *distance* between two vertices is the length (number of edges) of a shortest path between them. The distance $d(H_1, H_2)$ between two subgraphs H_1 and H_2 is the minimum of the distances between vertices $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$.

The well-known Four Color Theorem (Appel and Haken [3,4]) states that every planar graph is 4-colorable. Similarly, Grötzsch [8] proved that every triangle-free planar graph is 3-colorable. For some time, the question whether these results hold in the list coloring setting was open; finally, Voigt [14,15] found a planar graph that is not 4-choosable, and a triangle-free planar graph that is not 3-choosable. On the other hand, Thomassen [10,11] proved that every planar graph is 5-choosable

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[1] Supported by Institute for Theoretical Computer Science (ITI), project 1M0021620808 of Ministry of Education of Czech Republic, and by project GA201/09/0197 (Graph colorings and flows: structure and applications) of Czech Science Foundation.

0095-8956/$ – see front matter
http://dx.doi.org/10.1016/j.jctb.2013.10.004

and every planar graph of girth at least 5 is 3-choosable. Also, Kratochvíl and Tuza [9] observed that every planar triangle-free graph is 4-choosable.

Motivated by Grötzsch's result, Havel asked whether there exists a constant d such that if the distance between any two triangles in a planar graph is at least d, then the graph is 3-colorable. This question was open for many years, finally being answered in affirmative by Dvořák, Král' and Thomas [6] (although their bound on d is impractically large). Due to the result of Voigt [15], an analogous question for 3-choosability needs also to restrict 4-cycles: does there exist a constant d such that if the distance between any two ($\leqslant$4)-cycles in a planar graph is at least d, then the graph is 3-choosable? We give a positive answer to this question:

Theorem 1. *If G is a planar graph such that the distance between any two* ($\leqslant$4)*-cycles is at least* 26*, then G is* 3*-choosable.*

This bound is quite reasonable compared to one given for Havel's problem [6]. However, it is far from the best known lower bound of 4, given by Aksionov and Mel'nikov [2].

2. Proof of Theorem 1

For a subgraph H of a graph G, let $d(H) = \min_F d(H, F)$, where the minimum ranges over all ($\leqslant$4)-cycles F of G distinct from H. In the case that the graph G is not clear from the context, we write $d_G(H)$ instead. Let $t(G) = \min_H d(H)$, where the minimum ranges over all ($\leqslant$4)-cycles H of G. A *path of length k* (or a *k-path*) is a path with k edges and $k+1$ vertices. For a path or a cycle X, let $\ell(X)$ denote its length. Let r be the function defined by $r(0) = 0$, $r(1) = 2$, $r(2) = 4$, $r(3) = 9$, $r(4) = 13$ and $r(5) = 16$. For a path P of length at most 5, let $r(P) = r(\ell(P))$. Let $B = 26$.

A *relevant configuration* is a triple (H, Q, f), where H is a plane graph, Q is a subpath of the outer face of H and $f : V(H) \setminus V(Q) \to \{2, 3\}$ is a function. A graph G with list assignment L and a specified path P *contains* the relevant configuration (H, Q, f) if there exists an isomorphism π between H and a subgraph I of G (which we call the *image* of the configuration in G) such that π maps Q to $I \cap P$, and $|L(\pi(v))| = f(v)$ for every $v \in V(H) \setminus V(Q)$. Figs. 1 and 2 depict the relevant configurations that are used in the proof of Theorem 1 using the following conventions. The graph H is drawn in the figure. The path Q consists of the vertices drawn by full circles. The vertices to that f assigns value 2 are drawn by empty squares, and the vertices to that f assigns value 3 are drawn by empty circles.

Using the precoloring extension technique developed by Thomassen [11], we show the following generalization of Theorem 1:

Theorem 2. *Let G be a planar graph with the outer face C such that $t(G) \geqslant B$, and let P be a path such that $V(P) \subseteq V(C)$. Let L be a list assignment such that*

(S1) $|L(v)| = 3$ *for all* $v \in V(G) \setminus V(C)$;
(S2) $2 \leqslant |L(v)| \leqslant 3$ *for all* $v \in V(C) \setminus V(P)$;
(S3) $|L(v)| = 1$ *for all* $v \in V(P)$*, and the colors in the lists give a proper coloring of the subgraph of G induced by* $V(P)$;
(I) *the vertices with lists of size two form an independent set*;
(T) *if uvw is a triangle, $|L(u)| = 2$ and v has a neighbor with list of size two distinct from u, then w has no neighbor with list of size two distinct from u*; *and,*
(Q) *if a vertex v with list of size two has two neighbors w_1 and w_2 in P, then $L(v) \neq L(w_1) \cup L(w_2)$.*

Furthermore, assume that at least one of the following conditions is satisfied:

(OBSTa) $\ell(P) \leqslant 2$ *and all images in G of every relevant configuration drawn in Fig. 1 are L-colorable, or*
(OBSTb) $\ell(P) \leqslant 5$, $d(P) \geqslant r(P)$ *and all images in G of every relevant configuration drawn in Fig. 2 are L-colorable.*

Then G is L-colorable.