Journal of Combinatorial Theory, Series B 104 (2014) 60–71

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

The maximum number of complete subgraphs in a graph with given maximum degree

Jonathan Cutler [a], A.J. Radcliffe [b]

[a] Department of Mathematical Sciences, Montclair State University, 1 Normal Avenue, Montclair, NJ 07043, United States

[b] Department of Mathematics, University of Nebraska–Lincoln, Lincoln, NE 68588-0130, United States

ARTICLE INFO

Article history:
Received 7 June 2013
Available online 12 November 2013

Keywords:
Independent sets
Complete subgraphs
Extremal enumeration

ABSTRACT

Extremal problems involving the enumeration of graph substructures have a long history in graph theory. For example, the number of independent sets in a d-regular graph on n vertices is at most $(2^{d+1}-1)^{n/2d}$ by the Kahn–Zhao theorem [7,13]. Relaxing the regularity constraint to a minimum degree condition, Galvin [5] conjectured that, for $n \geqslant 2d$, the number of independent sets in a graph with $\delta(G) \geqslant d$ is at most that in $K_{d,n-d}$.
In this paper, we give a lower bound on the number of independent sets in a d-regular graph mirroring the upper bound in the Kahn–Zhao theorem. The main result of this paper is a proof of a strengthened form of Galvin's conjecture, covering the case $n \leqslant 2d$ as well. We find it convenient to address this problem from the perspective of $\overline{G}$. From this perspective, we show that the number of complete subgraphs of a graph G on n vertices with $\Delta(G) \leqslant r$, where $n = a(r+1)+b$ with $0 \leqslant b \leqslant r$, is bounded above by the number of complete subgraphs in $aK_{r+1} \cup K_b$.

1. Introduction

There has been quite a bit of recent interest in a range of extremal problems involving counting the number of a given type of substructure in a graph. For instance, the number of independent sets or the number of the complete subgraphs[1] of a graph. We let $\mathcal{I}(G)$ be the set of independent sets in the graph G and $\mathcal{K}(G)$ be the set of cliques in G. We write $i(G)$ and $k(G)$ for $|\mathcal{I}(G)|$ and $|\mathcal{K}(G)|$, respectively.

E-mail addresses: jonathan.cutler@montclair.edu (J. Cutler), aradcliffe1@math.unl.edu (A.J. Radcliffe).

[1] We use the term *clique* to refer to a complete subgraph, not necessarily a maximal complete subgraph.

0095-8956/$ – see front matter
http://dx.doi.org/10.1016/j.jctb.2013.10.003

A classic example of type of result we consider is the Kahn–Zhao theorem, proved first for bipartite graphs by Kahn [7] and later extended to all graphs by Zhao [13].

Theorem 1.1 *(Kahn–Zhao). If G is a d-regular graph with n vertices then*

$$i(G)^{\frac{1}{n}} \leqslant i(K_{d,d})^{\frac{1}{2d}} = \left(2^{d+1}-1\right)^{\frac{1}{2d}}.$$

The Kahn–Zhao theorem is tight when n is a multiple of $2d$ with $\frac{n}{2d}K_{d,d}$, i.e., $\frac{n}{2d}$ copies of $K_{d,d}$, achieving equality in the bound. Little is known about extremal examples when $2d$ does not divide n. One result of this paper is a corresponding theorem proving

$$i(G)^{1/n} \geqslant i(K_{d+1})^{1/(d+1)}$$

for G a d-regular graph on n vertices.

Extremal enumeration problems for complete subgraphs run in parallel to those for independent sets. If G is a graph with N independent sets, then $\overline{G}$ is a graph with N cliques. Any degree condition on G translates into a corresponding degree condition on $\overline{G}$. For example, the Kahn–Zhao theorem can be rephrased as follows: if G is an r-regular graph on n vertices, then

$$k(G)^{\frac{1}{n}} \leqslant k(2K_{n-1-r})^{\frac{1}{2(n-1-r)}}.$$

Note, however, that in the intuitively natural regime where r is fixed and n is large, we do not expect this bound to be tight.

Although regularity is a very natural condition to impose, a range of other conditions have been studied. For instance, it is a consequence of the Kruskal–Katona theorem [9,8] that among all graphs of given average degree, the lex graph[2] has the largest number of independent sets, indeed the largest number of independent sets of any fixed size. For a derivation, see, e.g., [2]. Another example is the oft-rediscovered result originally due to Zykov [14] (see also [4,11,6,10]) which bounds the number of cliques in graphs with bounded clique number, $\omega(G)$.

Theorem 1.2 *(Zykov). If G is a graph with n vertices and $\omega(G) \leqslant \omega$, then*

$$k(G) \leqslant k(T_{n,\omega}),$$

where $T_{n,\omega}$ is the Turán graph with ω parts.

Equivalently, this gives a bound on $i(G)$ for graphs with bounded independence number; the extremal graph is a union of disjoint complete graphs of almost equal sizes.

The main problem we discuss in this paper is that of computing, given n and r,

$$\max\bigl\{k(G)\colon\ G \text{ is a graph on } n \text{ vertices with } \Delta(G) \leqslant r\bigr\}.$$

This problem is equivalent to that of determining

$$\max\bigl\{i(G)\colon\ G \text{ is a graph on } n \text{ vertices with } \delta(G) \geqslant d\bigr\},$$

where $d = n-1-r$. Galvin [5] made the following conjecture.

Conjecture 1 *(Galvin). If G is a graph on n vertices with minimum degree at least d, where $n \geqslant 2d$, then $i(G) \leqslant i(K_{d,n-d})$.*

[2] The *lex graph with n vertices and m edges*, denoted $L(n,m)$, has vertex set $[n] = \{1,2,\dots,n\}$ and edge set an initial segment of size m in $\binom{[n]}{2}$ according to the lexicographic order.