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Defective 2-colorings of sparse graphs

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ABSTRACT

A graph G is (j,k)-colorable if its vertices can be partitioned into subsets V_1 and V_2 such that every vertex in $G[V_1]$ has degree at most j and every vertex in $G[V_2]$ has degree at most k. We prove that if $k \geqslant 2j+2$, then every graph with maximum average degree at most $2(2-\frac{k+2}{(j+2)(k+1)})$ is (j,k)-colorable. On the other hand, we construct graphs with the maximum average degree arbitrarily close to $2(2-\frac{k+2}{(j+2)(k+1)})$ (from above) that are not (j,k)-colorable. In fact, we prove a stronger result by establishing the best possible sufficient condition for the (j,k)-colorability of a graph G in terms of the minimum, $\varphi_{j,k}(G)$, of the difference $\varphi_{j,k}(W,G) = (2-\frac{k+2}{(j+2)(k+1)})|W| - |E(G[W])|$ over all subsets W of $V(G)$. Namely, every graph G with $\varphi_{j,k}(G) > \frac{-1}{k+1}$ is (j,k)-colorable. On the other hand, we construct infinitely many non-(j,k)-colorable graphs G with $\varphi_{j,k}(G) = \frac{-1}{k+1}$.

1. Introduction

A graph G is called *improperly* $(d_1,\dots,d_k)$*-colorable*, or just $(d_1,\dots,d_k)$*-colorable*, if the vertex set of G can be partitioned into subsets $V_1,\dots,V_k$ such that the graph $G[V_i]$ induced by the vertices of V_i has maximum degree at most d_i for all $1 \leqslant i \leqslant k$. This notion generalizes those of proper k-coloring (when $d_1 = \dots = d_k = 0$) and d-improper k-coloring (when $d_1 = \dots = d_k = d \geqslant 1$).

Proper and d-improper colorings have been widely studied. As shown by Appel and Haken [1,2], every planar graph is 4-colorable, i.e. $(0,0,0,0)$-colorable. Cowen, Cowen and Woodall [8] proved that

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every planar graph is 2-improperly 3-colorable, i.e. $(2, 2, 2)$-colorable. This latter result was extended by Havet and Sereni [10] to sparse graphs that are not necessarily planar: for every $k \geqslant 0$, every graph G with $\mathrm{mad}(G) < \frac{4k+4}{k+2}$ is k-improperly 2-colorable, i.e. (k, k)-colorable.

Recall that for a graph G, $\mathrm{mad}(G) = \max\{\frac{2|E(H)|}{|V(H)|}, H \subseteq G\}$ is the maximum over the average degrees of the subgraphs of G. The *girth*, $g(G)$, of a graph G is the length of a shortest cycle in G.

We will consider probably the simplest version of defective colorings, defective colorings with two colors. For nonnegative integers j and k, let $F(j, k)$ denote the supremum of x such that every graph G with $\mathrm{mad}(G) \leqslant x$ is (j, k)-colorable. It is easy to see that $F(0, 0) = 2$. Indeed, since the odd cycle C_{2n-1} has $\mathrm{mad}(G) = 2$ and is not $(0, 0)$-colorable, $F(0, 0) \leqslant 2$. On the other hand, each graph with $\mathrm{mad}(G) < 2$ has no cycles and therefore is bipartite, i.e., $(0, 0)$-colorable.

Glebov and Zambalaeva [9] proved that every planar graph G with $g(G) \geqslant 16$ is $(0, 1)$-colorable. This was strengthened by Borodin and Ivanova [3] by proving that every graph G with $\mathrm{mad}(G) < \frac{7}{3}$ is $(0, 1)$-colorable, which implies that every planar graph G with $g(G) \geqslant 14$ is $(0, 1)$-colorable. In [4], it was proved that $F(0, 1) = \frac{12}{5}$. In particular, this implies that every planar graph G with $g(G) \geqslant 12$ is $(0, 1)$-colorable.

For each integer $k \geqslant 2$, Borodin et al. [5] proved that every graph G with $\mathrm{mad}(G) < \frac{3k+4}{k+2} = 3 - \frac{2}{k+2}$ is $(0, k)$-colorable. On the other hand, for all $k \geqslant 2$ Borodin et al. [5] constructed non-$(0, k)$-colorable graphs with mad arbitrarily close to $\frac{3k+2}{k+1} = 3 - \frac{1}{k+1}$.

Recently, it was proved by Borodin et al. [7] that every graph G with $\mathrm{mad}(G) < \frac{10k+22}{3k+9}$, where $k \geqslant 2$, is $(1, k)$-colorable. On the other hand, [7] presents a construction of non-$(1, k)$-colorable graphs whose maximum average degree is arbitrarily close to $\frac{14k}{4k+1}$.

The purpose of this paper is to prove an exact result for a wide range of j and k.

Theorem 1. *Let*

$$j \geqslant 0 \quad \textit{and} \quad k \geqslant 2j + 2. \tag{1}$$

Then $F(j, k) = 2(2 - \frac{k+2}{(j+2)(k+1)})$.

In particular, together with [4], Theorem 1 yields exact values for $F(0, k)$ for every k. If $j \leqslant k < 2j + 2$, then we do not know the exact answer apart from the cases $j = 0$ and $k \in \{0, 1\}$. Furthermore, the formula for $F(0, 1)$ differs from that in Theorem 1.

In fact, to derive Theorem 1, we will need a more precise statement. For a graph G and $W \subseteq V(G)$, let

$$\varphi_{j,k}(W, G) := \left(2 - \frac{k+2}{(j+2)(k+1)}\right)|W| - \bigl|E\bigl(G[W]\bigr)\bigr|. \tag{2}$$

By definition, $\mathrm{mad}(G) \leqslant 2(2 - \frac{k+2}{(j+2)(k+1)})$ if and only if $\varphi_{j,k}(W, G) \geqslant 0$ for every $W \subseteq V(G)$.

Theorem 2. *Let j and k satisfy (1). Every graph G such that*

$$\varphi_{j,k}(W, G) > -\frac{1}{k+1} \quad \textit{for every } W \subseteq V(G), \tag{3}$$

is (j, k)-colorable. Moreover, restriction (3) is sharp.

The second part of Theorem 2 means that there exist infinitely many non-(j, k)-colorable graphs G for which the non-strict version of (3) holds.

Since each planar graph G satisfies $\mathrm{mad}(G) < \frac{2g(G)}{g(G)-2}$, from Theorem 2 we easily deduce:

Corollary 1. *Let G be a planar graph and*

$$k \geqslant \max\left\{-1 + \frac{g(G) - 2}{(g(G) - 4)(j + 2) - g(G) + 2}, 2j + 2\right\}.$$