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Excluding pairs of graphs

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ABSTRACT

For a graph G and a set of graphs $\mathcal{H}$, we say that G is $\mathcal{H}$*-free* if no induced subgraph of G is isomorphic to a member of $\mathcal{H}$. Given an integer $P > 0$, a graph G, and a set of graphs $\mathcal{F}$, we say that G *admits an* $(\mathcal{F}, P)$*-partition* if the vertex set of G can be partitioned into P subsets $X_1, \dots, X_P$, so that for every $i \in \{1, \dots, P\}$, either $|X_i| = 1$, or the subgraph of G induced by X_i is $\{F\}$-free for some $F \in \mathcal{F}$.
Our first result is the following. For every pair (H, J) of graphs such that H is the disjoint union of two graphs H_1 and H_2, and the complement J^c of J is the disjoint union of two graphs J_1^c and J_2^c, there exists an integer $P > 0$ such that every $\{H, J\}$-free graph has an $(\{H_1, H_2, J_1, J_2\}, P)$-partition. A similar result holds for tournaments, and this yields a short proof of one of the results of [1].
A *cograph* is a graph obtained from single vertices by repeatedly taking disjoint unions and disjoint unions in the complement. For every cograph there is a parameter measuring its complexity, called its *height*. Given a graph G and a pair of graphs H_1, H_2, we say that G is $\{H_1, H_2\}$*-split* if $V(G) = X_1 \cup X_2$, where the subgraph of G induced by X_i is $\{H_i\}$-free for $i = 1, 2$. Our second result is that for every integer $k > 0$ and pair $\{H, J\}$ of cographs each of height $k+1$, where neither of H, J^c is connected, there exists a pair of cographs $(\tilde{H}, \tilde{J})$, each of height k, where neither of $\tilde{H}^c, \tilde{J}$ is connected, such that every $\{H, J\}$-free graph is $\{\tilde{H}, \tilde{J}\}$-split.
Our final result is a construction showing that if $\{H, J\}$ are graphs each with at least one edge, then for every pair of integers r, k there exists a graph G such that every r-vertex

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induced subgraph of G is $\{H, J\}$-split, but G does not admit an $(\{H, J\}, k)$-partition.

1. Introduction

All graphs in this paper are finite and simple. Let G be a graph. For $X \subseteq V(G)$, we denote by $G|X$ the subgraph of G induced by X. The complement of G, denoted by G^c, is the graph with vertex set $V(G)$ such that two vertices are adjacent in G if and only if they are non-adjacent in G^c. A *clique* in G is a set of vertices all pairwise adjacent; and a *stable set* in G is a set of vertices all pairwise non-adjacent. For disjoint $X, Y \subseteq V(G)$, we say that X is *complete* (*anticomplete*) to Y if every vertex of X is adjacent (non-adjacent) to every vertex of Y. If $|X| = 1$, say $X = \{x\}$, we say "x is complete (anticomplete) to Y" instead of "$\{x\}$ is complete (anticomplete) to Y".

We denote by K_n the complete graph on n vertices, and by S_n the complement of K_n. A graph is *complete multipartite* if its vertex set can be partitioned into stable sets, all pairwise complete to each other. For graphs H and G we say that G *contains* H if some induced subgraph of G is isomorphic to H. Let $\mathcal{F}$ be a set of graphs. We say that G is $\mathcal{F}$*-free* if G contains no member of $\mathcal{F}$. If $|\mathcal{F}| = 1$, say $\mathcal{F} = \{F\}$, we write "G is F-free" instead of "G is $\{F\}$-free". For a pair of graphs $\{H_1, H_2\}$, we say that G is $\{H_1, H_2\}$*-split* if $V(G) = X_1 \cup X_2$, and $G|X_i$ is H_i-free for $i = 1, 2$. We remind the reader that a *split graph* is a graph whose vertex set can be partitioned into a clique and a stable set; thus in our language split graphs are precisely the graphs that are $\{K_2, S_2\}$-split.

Ramsey's theorem can be restated in the following way: for every pair of integers $m, n > 0$ there exists an integer P, such that for every $\{S_m, K_n\}$-free graph G, $V(G)$ can be partitioned into at most P well-understood parts (in fact, each part is a single vertex). One might ask whether a similar statement holds for more general pairs of graphs than just $\{S_m, K_n\}$ (adjusting the definition of "well-understood").

For instance, a result of [3] implies that if G is $\{C_4, C_4^c\}$-free (where C_4 is a cycle on four vertices), then $V(G)$ can be partitioned into three parts, each of which induces either a complete graph, or a graph with no edges, or a cycle of length five.

In [2] two of us made progress on this question, but to state the result we first need a definition. Given an integer $P > 0$, we say that a graph G *admits an* $(\mathcal{F}, P)$*-partition* if $V(X) = X_1 \cup \cdots \cup X_P$ such that for every $i \in \{1, \ldots, P\}$, either $|X_i| = 1$ or $G|X_i$ is $\{F\}$-free for some $F \in \mathcal{F}$. Please note that the first alternative in the definition of an $(\mathcal{F}, P)$-partition (the condition that $|X_i| = 1$) is only necessary when no graph in $\mathcal{F}$ has more than one vertex. We proved the following:

1.1. *For every pair of graphs (H, J) such that H^c and J are complete multipartite, there exist integers $k, P > 0$ such that every $\{H, J\}$-free graph admits a $(\{K_k, S_k\}, P)$-partition.*

and its immediate corollary: