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Primitive groups synchronize non-uniform maps of extreme ranks

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ABSTRACT

Let Ω be a set of cardinality n, G a permutation group on Ω, and $f : \Omega \to \Omega$ a map which is not a permutation. We say that G synchronizes f if the semigroup $\langle G, f\rangle$ contains a constant map.

The first author has conjectured that a primitive group synchronizes any map whose kernel is non-uniform. Rystsov proved one instance of this conjecture, namely, degree n primitive groups synchronize maps of rank $n-1$ (thus, maps with kernel type $(2, 1, \ldots, 1)$). We prove some extensions of Rystsov's result, including this: a primitive group synchronizes every map whose kernel type is $(k, 1, \ldots, 1)$. Incidentally this result provides a new characterization of imprimitive groups. We also prove that the conjecture above holds for maps of extreme ranks, that is, ranks 3, 4 and $n-2$.

These proofs use a graph-theoretic technique due to the second author: a transformation semigroup fails to contain a constant map if and only if it is contained in the endomorphism semigroup of a non-null (simple undirected) graph.

The paper finishes with a number of open problems, whose solutions will certainly require very delicate graph theoretical considerations.

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1. Introduction

In automata theory, the well-known Černý conjecture states that a synchronizing automaton with n states has a synchronizing word of length $(n-1)^2$. (For many references on the growing bibliography on this problem please see the two websites [18,21].) Solving this conjecture is equivalent to proving that given a set $S = \{f_1, \ldots, f_m\}$ of transformations on a finite set $\Omega := \{1, \ldots, n\}$, if S generates a constant, then S generates a constant in a length $(n-1)^2$ word on its generators. This conjecture has been established when $\langle S \rangle$ is a semigroup in which all its subgroups are trivial [22]. So it remains to prove the conjecture for semigroups containing non trivial subgroups; the case in which the semigroup contains a permutation group is a particular instance of this general problem. In addition, the known examples witnessing the optimality of the Černý bound contain a permutation among the given set of generators S, so they make it especially interesting to study the cases in which a subset of S generates a permutation group.

Let G be a permutation group on a set Ω with $|\Omega| = n$. We say that G *synchronizes* a map f on Ω if the semigroup $\langle G, f \rangle$ contains a constant map. G is said to be *synchronizing* if G synchronizes every non-invertible transformation on Ω. The *diameter* of a group is the largest diameter of its Cayley graphs. Taking into account the motivation of the considerations above, the ultimate goal is to find a classification of the synchronizing groups and then study those with the largest diameter, since they should assist the generation of a constant with the lowest diligence. But even when we forget about the automata motivation of these problems, the classification of synchronizing groups (a class strictly between primitivity and 2-homogeneity) and the study of their diameters are very interesting questions in themselves, as well as extremely demanding (please see [4,8,9,17,16]).

Recall that the *rank* of map f on Ω is $|\Omega f|$, and the *kernel* of f is the partition of Ω into the inverse images of points in the image of f; equivalently, the kernel of f is the partition of Ω induced by the equivalence relation $\{(x, y) \in \Omega \times \Omega \mid xf = yf\}$. The *kernel type* of f is the partition of n given by the sizes of the parts of the kernel. A partition of Ω is *uniform* if all its parts have the same size. We will call a map *uniform* if its kernel is uniform.

We note that, if a transformation semigroup S contains a transitive group G but not a constant function, then the image I of a map f of minimal rank in S is a G*-section* for the kernel of f, in the sense that Ig is a section for $\ker(f)$, for all $g \in G$; in addition, the map f has uniform kernel (see Neumann [17]).

In [4] the conjecture that a primitive group of permutations of Ω synchronizes every non-uniform transformation on Ω was proposed. In 1995 Rystsov [19] proved the following particular instance of this conjecture.

Theorem 1. *A transitive permutation group G of degree n is primitive if and only if it synchronizes every map of rank $n-1$.*

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