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Exactly m-coloured complete infinite subgraphs

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ABSTRACT

Given an edge colouring of a graph with a set of m colours, we say that the graph is *exactly m-coloured* if each of the colours is used. The question of finding exactly m-coloured complete subgraphs was first considered by Erickson in 1994; in 1999, Stacey and Weidl partially settled a conjecture made by Erickson and raised some further questions. In this paper, we shall study, for a colouring of the edges of the complete graph on $\mathbb{N}$ with exactly k colours, how small the set of natural numbers m for which there exists an exactly m-coloured complete infinite subgraph can be. We prove that this set must have size at least $\sqrt{2k}$; this bound is tight for infinitely many values of k. We also obtain a version of this result for colourings that use infinitely many colours.

1. Introduction

A classical result of Ramsey [10] says that when the edges of a complete graph on a countably infinite vertex set are finitely coloured, one can always find a complete infinite subgraph all of whose edges have the same colour.

Ramsey's theorem has since been generalised in many ways; most of these generalisations are concerned with finding other monochromatic structures. For a survey of many of these generalisations, see the book of Graham, Rothschild and Spencer [8]. Ramsey

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theory has witnessed many developments over the last fifty years and continues to be an area of active research today; see, for instance, [9,1,13,2].

Alternatively, anti-Ramsey theory, which originates in a paper of Erdős, Simonovits and Sós [5], is concerned with finding large "rainbow coloured" or "totally multicoloured" structures. Between these two ends of the spectrum, one could consider the question of finding structures which are coloured with exactly m different colours as was first done by Erickson [6]; it is this line of enquiry that we pursue here.

2. Our results

For a set X, denote by $X^{(2)}$ the set of all unordered pairs of elements of X; equivalently, $X^{(2)}$ is the complete graph on the vertex set X. As usual, $[n]$ will denote $\{1,\dots,n\}$, the set of the first n natural numbers. By a *colouring* of a graph G, we will always mean a colouring of the edges of G.

Let $\Delta : \mathbb{N}^{(2)} \twoheadrightarrow [k]$ be a surjective k-colouring of the edges of the complete graph on the natural numbers with $k \geqslant 2$ colours. We say that a subset $X \subset \mathbb{N}$ is (*exactly*) *m-coloured* if $\Delta(X^{(2)})$, the set of values attained by Δ on the edges with both endpoints in X, has size exactly m. Our aim in this paper is to study the set

$$\mathcal{F}_\Delta := \bigl\{m \in [k]\colon\ \exists X \subset \mathbb{N} \text{ such that } X \text{ is infinite and } m\text{-coloured}\bigr\}.$$

Clearly, $k \in \mathcal{F}_\Delta$ as Δ is surjective. Ramsey's theorem tells us that $1 \in \mathcal{F}_\Delta$. Furthermore, Erickson [6] noted that a fairly straightforward application of Ramsey's theorem enables one to show that $2 \in \mathcal{F}_\Delta$ for any surjective k-colouring Δ with $k \geqslant 2$. He also conjectured that with the exception of $1, 2$ and k, no other elements are guaranteed to be in $\mathcal{F}_\Delta$ and that if $k > k' > 2$, then there is a surjective k-colouring Δ such that $k' \notin \mathcal{F}_\Delta$. Stacey and Weidl [11] settled this conjecture in the case where k is much bigger than k'. More precisely, for any $k' > 2$, they showed that there is a constant $C_{k'}$ such that if $k > C_{k'}$, then there is a surjective k-colouring Δ such that $k' \notin \mathcal{F}_\Delta$.

In this note, we shall be interested in the set of possible sizes of $\mathcal{F}_\Delta$. Since $\mathcal{F}_\Delta \subset [k]$, we have $|\mathcal{F}_\Delta| \leqslant k$ and it is easy to see that equality is in fact possible. Things are not so clear when we turn to the question of lower bounds. Let us define

$$\psi(k) := \min_{\Delta:\mathbb{N}^{(2)} \twoheadrightarrow [k]} |\mathcal{F}_\Delta|.$$

We are able to prove the following lower bound for $\psi(k)$.

Theorem 1. *Let $n \geqslant 2$ be the largest natural number such that $k \geqslant \binom{n}{2}+1$. Then $\psi(k) \geqslant n$.*

It is not hard to check that Theorem 1 is tight when $k = \binom{n}{2} + 1$ for some $n \geqslant 2$. To this end, we consider the "small-rainbow colouring" Δ which colours all the edges with both endpoints in $[n]$ with $\binom{n}{2}$ distinct colours and all the remaining edges with the one