Journal of Combinatorial Theory, Series B 103 (2013) 767–794

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Coloring simple hypergraphs

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ARTICLE INFO

Article history:
Received 14 November 2011
Available online 14 October 2013

Keywords:
Hypergraph coloring
Semi-random method

ABSTRACT

Fix an integer $k \geqslant 3$. A k-uniform hypergraph is simple if every two edges share at most one vertex. We prove that there is a constant c depending only on k such that every simple k-uniform hypergraph H with maximum degree Δ has chromatic number satisfying

$$\chi(H) < c\left(\frac{\Delta}{\log \Delta}\right)^{\frac{1}{k-1}}.$$

This implies a classical result of Ajtai, Komlós, Pintz, Spencer and Szemerédi and its strengthening due to Duke, Lefmann and Rödl. The result is sharp apart from the constant c.

1. Introduction

Hypergraph coloring has been studied for almost 50 years, since Erdős' seminal results on the minimum number of edges in uniform hypergraphs that are not 2-colorable. Some of the major tools in combinatorics have been developed to solve problems in this area, for example, the Local Lemma, [7] and the nibble or semi-random method. Consequently, the subject enjoys a prominent place among basic combinatorial questions.

Closely related to coloring problems are questions about the independence number of hypergraphs. An easy extension of Turán's graph theorem shows that a k-uniform hypergraph with n vertices and average degree d has an independent set of size at least $cn/d^{1/(k-1)}$, where c depends only on k. If we impose local constraints on the hypergraph, then this bound can be improved. An i-cycle in

[1] Supported in part by NSF Grants DMS-0753472 and CCF-1013110.
[2] Supported in part by NSF Grants DMS-0653946, 0969092, 1300138.

0095-8956/$ – see front matter
http://dx.doi.org/10.1016/j.jctb.2013.09.003

a k-uniform hypergraph is a collection of i distinct edges spanned by at most $i(k-1)$ vertices. Say that a k-uniform hypergraph has girth at least g if it contains no i-cycles for $2 \leqslant i < g$. Call a k-uniform hypergraph simple if it has girth at least 3. In other words, every two edges have at most one vertex in common. Throughout this paper we will assume that $k \geqslant 3$ is a fixed positive integer.

Ajtai, Komlós, Pintz, Spencer and Szemerédi [2] proved the following fundamental result that strengthened the bound obtained by Turán's theorem above.

Theorem 1. *(See [2].) Let $H = (V, E)$ be a k-uniform hypergraph of girth at least 5 with maximum degree Δ. Then it has an independent set of size at least*

$$cn\left(\frac{\log \Delta}{\Delta}\right)^{1/(k-1)}$$

where c depends only on k.

Spencer conjectured that Theorem 1 holds even for simple hypergraphs, and this was later proved by Duke, Lefmann and Rödl [6]. Theorem 1 has proved to be a seminal result in combinatorics, with many applications. Indeed, the result was first proved for $k = 3$ by Komlós, Pintz and Szemerédi [13] to disprove the famous Heilbronn conjecture, that among every set of n points in the unit square, there are three points that form a triangle whose area is at most $O(1/n^2)$. For applications of Theorem 1 to coding theory or combinatorics, see [15] or [14], respectively.

The goal of this paper is to prove a result that is stronger than Theorem 1 (and also the accompanying result of [6]). Since the proof of our result does not use Theorem 1, it gives an alternative proof of all the applications of Theorem 1 as well. Our main result states not only that one can find an independent set of the size guaranteed by Theorem 1, but also that the entire vertex set can be partitioned into independent sets with the average size as in Theorem 1. Recall that the chromatic number $\chi(H)$ of H is the minimum number of colors needed to color the vertex set so that no edge is monochromatic.

Theorem 2. *Fix $k \geqslant 3$. Let $H = (V, E)$ be a simple k-uniform hypergraph with maximum degree Δ. Then*

$$\chi(H) < c\left(\frac{\Delta}{\log \Delta}\right)^{\frac{1}{k-1}}$$

where c depends only on k.

It is shown in [5] that Theorem 2 is sharp apart from the constant c. In order to prove Theorem 2 we will first prove the following slightly weaker result. A triangle in a k-uniform hypergraph is a 3-cycle that contains no 2-cycle. In other words, it is a collection of three sets A, B, C such that every two of these sets have intersection of size one, and $A \cap B \cap C = \emptyset$.

Theorem 3. *Fix $k \geqslant 3$. Let $H = (V, E)$ be a simple triangle-free k-uniform hypergraph with maximum degree Δ. Then*

$$\chi(H) < c\left(\frac{\Delta}{\log \Delta}\right)^{\frac{1}{k-1}},$$

where c depends only on k.

The proof of Theorem 3 rests on several major developments in probabilistic combinatorics over the past 25 years. Our approach is inspired by Johansson's [9] breakthrough result on graph coloring, which proved Theorem 3 for $k = 2$.

The proof technique, which has been termed the semi-random, or nibble method, was first used by Rödl [17] (inspired by earlier work in [2,13]) to confirm the Erdős–Hanani conjecture about the