Journal of Combinatorial Theory, Series B 102 (2012) 832–838

Contents lists available at SciVerse ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Independent sets in direct products of vertex-transitive graphs

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ARTICLE INFO

Article history:
Received 5 July 2010
Available online 30 December 2011

Keywords:
Direct product
Primitivity
Independence number
Vertex-transitive

ABSTRACT

The direct product $G \times H$ of graphs G and H is defined by

$$V(G \times H) = V(G) \times V(H)$$

and

$$E(G \times H) = \bigl\{\bigl[(u_1, v_1), (u_2, v_2)\bigr]:\ (u_1, u_2) \in E(G) \text{ and } (v_1, v_2) \in E(H)\bigr\}.$$

In this paper, we will prove that

$$\alpha(G \times H) = \max\bigl\{\alpha(G)|H|, \alpha(H)|G|\bigr\}$$

holds for all vertex-transitive graphs G and H, which provides an affirmative answer to a problem posed by Tardif (1998) [11]. Furthermore, the structure of all maximum independent sets of $G \times H$ is determined.

1. Introduction

Let G and H be two graphs. The direct product $G \times H$ of G and H is defined by

$$V(G \times H) = V(G) \times V(H)$$

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[1] Partially supported by the National Natural Foundation of China (No. 11001249) and Opening Fund of Top Key Discipline of Computer Software and Theory in Zhejiang Provincial Colleges at Zhejiang Normal University.

0095-8956/$ – see front matter
doi:10.1016/j.jctb.2011.12.005

and

$$E(G \times H) = \{[(u_1, v_1), (u_2, v_2)]:\ (u_1, u_2) \in E(G) \text{ and } (v_1, v_2) \in E(H)\}.$$

It is easy to see this product is commutative and associative, and the product of more than two graphs is well defined. For a graph G, the products $G^n = G \times G \times \cdots \times G$ is called the n-th power of G.

An interesting problem is the independence number of $G \times H$. It is clear that if I is an independent set of G or H, then the preimage of I under projections is an independent set of $G \times H$, and so $\alpha(G \times H) \geqslant \max\{\alpha(G)|H|, \alpha(H)|G|\}$. Here $|G|$ denotes the order of G, i.e., $|V(G)|$. It is natural to ask whether the equality holds or not. In general, the equality does not hold for non-vertex-transitive graphs (see [7]). So Tardif [11] posed the following problem.

Problem 1.1. (See Tardif [11].) Does the equality

$$\alpha(G \times H) = \max\{\alpha(G)|H|, \alpha(H)|G|\}$$

hold for all vertex-transitive graphs G and H?

Furthermore, it immediately raises another interesting problem:

Problem 1.2. When $\alpha(G \times H) = \max\{\alpha(G)|H|, \alpha(H)|G|\}$, is every maximum independent set of $G \times H$ the preimage of an independent set of one factor under projections?

If the answer to Problem 1.2 is yes, we then say the direct product $G \times H$ is *MIS-normal* (maximum-independent-set-normal). Furthermore, the direct product $G_1 \times G_2 \times \cdots \times G_n$ is said to be MIS-normal if every maximum independent set of it is the preimage of an independent set of one factor under projections.

The two problems have received some attention. Frankl [6] and Valencia-Pabon and Vera [12] solved Problem 1.1 for Kneser graphs and circular graphs, respectively. Ahlswede et al. [1] generalized Frankl's results. Ku and Wong [9] investigated the structure of maximum independent sets in direct products of permutation graphs; Wang and Yu [13] proved that both Problems 1.1 and 1.2 have positive answers if one of G and H is a bipartite graph. Larose and Tardif [10] investigated the structures of maximum independent sets in powers of circular graphs, Kneser graphs and truncated simplices. For an arbitrary vertex-transitive graph G, they asked whether or not G^n is MIS-normal for all $n \geqslant 2$ if G^2 is MIS-normal. This question has been answered positively independently by Ku and McMillan [8] and the author [15].

Given a graph G and a real number r, a *fractional r-coloring* of G is a mapping f which assigns to each independent set I of G a real number $f(I) \in [0, 1]$ so that $\sum f(I) = r$ and for any vertex v, $\sum_{v \in I} f(I) \geqslant 1$. The *fractional chromatic number* $\chi_f(G)$ of G is the minimum r such that G has a fractional r-coloring. It is well known that if G is a vertex transitive graph, then $\chi_f(G) = |V(G)|/\alpha(G)$. A generalization of Problem 1.1 is studied in [16], where the following question is asked: Is it true that for any graphs G and H, $\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$? After the original version of this paper, this question was answered positively in [17], which implies a positive solution to Problem 1.1.

In this paper we shall solve both Problem 1.1 and Problem 1.2. To state our results we need to introduce some notations and notions.

For a graph G, let $I(G)$ denote the set of all maximum independent sets of G. Given a subset A of $V(G)$, we define

$$N_G(A) = \{b \in V(G):\ (a, b) \in E(G) \text{ for some } a \in A\},$$

$$N_G[A] = N_G(A) \cup A \quad \text{and} \quad \overline{N}_G[A] = V(G) - N_G[A].$$

If G is clear from the context, for simplicity, we will omit the index G.

In [15], by the so-called "No-Homomorphism" lemma of Albertson and Collins [2] we proved the following result.