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Non-zero disjoint cycles in highly connected group labelled graphs

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Abstract

Let $G = (V, E)$ be an oriented graph whose edges are labelled by the elements of a group Γ. A cycle C in G has non-zero weight if for a given orientation of the cycle, when we add the labels of the forward directed edges and subtract the labels of the reverse directed edges, the total is non-zero. We are specifically interested in the maximum number of vertex disjoint non-zero cycles.

We prove that if G is a Γ-labelled graph and $\overline{G}$ is the corresponding undirected graph, then if $\overline{G}$ is $\frac{31}{2}k$-connected, either G has k disjoint non-zero cycles or it has a vertex set Q of order at most $2k-2$ such that $G - Q$ has no non-zero cycles. The bound "$2k-2$" is best possible.

This generalizes the results due to Thomassen (The Erdős–Pósa property for odd cycles in graphs with large connectivity, Combinatorica 21 (2001) 321–333.), Rautenbach and Reed (The Erdős–Pósa property for odd cycles in highly connected graphs, Combinatorica 21 (2001) 267–278.) and Kawarabayashi and Reed (Highly parity linked graphs, preprint.), respectively.

MSC: 05C40; 05C83

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1. Introduction

A family $\mathcal{F}$ of graphs has the *Erdős–Pósa property*, if for every integer k there exists an integer $f(k, \mathcal{F})$ such that every graph G contains either k vertex-disjoint subgraphs each isomorphic to

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a graph in $\mathcal{F}$ or a set C of at most $f(k, \mathcal{F})$ vertices such that $G - C$ has no subgraph isomorphic to a graph in $\mathcal{F}$. The term *Erdős–Pósa property* arose because in [5], Erdős and Pósa proved that the family of cycles has this property.

The situation is different when we consider the family of odd cycles. Lovàsz characterizes the graphs having no two disjoint odd cycles, using Seymour's result on regular matroids. No such characterization is known for more than three odd cycles. In fact, the Erdős–Pósa property does not hold for odd cycles. Reed [13] observed that there exists a cubic projective planar graph which does not contain two edge-disjoint odd cycles, but there is neither a vertex set A nor an edge set B of a bounded cardinality such that $G - A$ and $G - B$ are bipartite. In fact, this example shows that the Erdős–Pósa property does not necessarily hold for any cycle of length $\not\equiv 0$ modulo m, see [16].

While the Erdős–Pósa property does not hold for odd cycles in general, Reed [13] proved that the Erdős–Pósa property holds for odd cycles in planar graphs. This result was extended to an orientable fixed surface in [9]. Note that the Erdős–Pósa property does not hold for odd cycles in non-orientable surfaces, even for projective planar graphs as the above example shows. But such an example on the projective plane or on the non-orientable surface is not 5-connected, so one can hope that if a graph is highly connected compared to k, then the Erdős–Pósa property holds for odd cycles. Motivated by this, Thomassen [17] was the first to prove that there exists a function $f(k)$ such that every $f(k)$-connected graph G has either k disjoint odd cycles or a vertex set X of order at most $2k - 2$ such that $G - X$ is bipartite. Hence, he showed that the Erdős–Pósa property holds for odd cycles in highly connected graphs. Soon after that, Rautenbach and Reed [12] proved that the connectivity $f(k) \geqslant 576k$ will do the same. Very recently, Kawarabayashi and Reed [10] further improved to the connectivity $f(k) \geqslant 24k$. The bound "$2k - 2$" is best possible in a sense since a large bipartite graph with edges of a complete graph of $2k - 1$ vertices added in one partite set shows that no matter how large the connectivity is, there are no k disjoint odd cycles.

In this paper, we are interested in group labelled graphs. Let Γ be an arbitrary group. We will use additive notation for groups, though they need not be abelian. Let G be an oriented graph. For each edge e in G, we assign a weight γ_e. The weight γ_e is added when the edge is traversed according to the orientation and subtracted when traversed contrary to the orientation. Rigorously, given an oriented graph G and a group Γ, a Γ-*labelling of* G consists of an assignment of a label γ_e to every edge $e \in E$, and function $\gamma : \{(e, v) | e \in E(G), v \text{ an end of } e\} \to \Gamma$ such that for every edge $e = (u, v)$ in G, where u is the tail of e and v is the head, $\gamma(e, u) = -\gamma_e = -\gamma(e, v)$. Let $C = (v_0 e_1 v_1 e_2 \dots e_k v_k = v_0)$ be a (not necessarily directed) cycle in G. Then the weight of C, denoted by $w(C)$, is $\sum_{i=1}^{k} \gamma(e_i, v_i)$. While the weight of a cycle will generally depend upon the orientation in which we traverse the edges and the vertex chosen to be v_0, we will in general only be concerned whether or not the weight of a particular cycle is non-zero. This is independent of the orientation of the cycle or the initial vertex.

Our main theorem is the following.

Theorem 1.1. *Let G be an oriented graph and Γ a group. Let the function γ be a Γ labelling of G. Let $\overline{G}$ be the underlying undirected graph. If $\overline{G}$ is $\frac{31}{2}k$-connected, then G has either k disjoint non-zero cycles or it has a vertex set Q of order at most $2k - 2$ such that $G - Q$ has no non-zero cycles.*

This generalizes the results due to Thomassen [17], Rautenbach and Reed [12] and Kawarabayashi and Reed [10], respectively. Given a graph G, assign edge directions arbitrarily,