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Note

On the expansion rate of Margulis expanders

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Abstract

In this note we determine exactly the expansion rate of an infinite 4-regular expander graph which is a variant of an expander due to Margulis. The vertex set of this graph consists of all points in the plane. The point (x, y) is adjacent to the points $S(x, y)$, $S^{-1}(x, y)$, $T(x, y)$, $T^{-1}(x, y)$ where $S(x, y) = (x, x + y)$ and $T(x, y) = (x + y, y)$. We show that the expansion rate of this 4-regular graph is 2. The main technical result asserts that for any compact planar set A of finite positive measure,

$$\frac{|S(A) \cup S^{-1}(A) \cup T(A) \cup T^{-1}(A) \cup A|}{|A|} \geqslant 2,$$

where $|B|$ is the Lebesgue measure of B.

The proof is completely elementary and is based on *symmetrization*—a classical method in the area of isoperimetric problems. We also use symmetrization to prove a similar result for a directed version of the same graph.

Keywords: Margulis expanders; Expansion rate; Symmetrization

1. Introduction

The Greek *isoperimetric problem* asks for the largest possible area of a planar figure of a given circumference. It took about two millennia to prove that everyone's guess is true: the optimal figure is the disk. Modern proofs for this fact are pretty easy, but perhaps the most conceptual proofs known are based on the notion of *symmetrization*. The basic idea is this: given any planar figure K, we seek a "more symmetric" figure K' of the same circumference, so that $|K'| \geqslant |K|$ (where $|X|$ is the Lebesgue measure of X). After the appropriate symmetries are identified, one

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shows that the optimum is attained for figures that are invariant under all relevant symmetries. In the planar case, the relevant symmetries can be taken to be reflections with respect to lines through the set's center of gravity. In this case the disk is the unique invariant set which is, therefore, also the unique optimal body. As the reader probably knows, we are telling here only part of the story that is relevant to us, and some additional argumentation is needed to complete the proof.

This classical problem is the starting point for a lot of modern mathematics. Specifically, it is often more challenging to answer similar questions for underlying geometries other than the Euclidean plane. Indeed, the theory of expander graphs can be viewed as a modern discrete version of this classical problem. Here we attempt to deal with the problem of presenting a family of expander graphs, and of proving their expansion properties by calculating their *exact* expansion rate. The expanders under consideration were the first to have been explicitly constructed, and are due to Margulis ([4],1973). Technically, we determine the expansion rate of a variant of his infinite 4-regular graph.

In his work, Margulis relied on deep theorems from the theory of groups representations of the group $SL_2(Z_p)$. He used five transformations that generated the associated affine group (namely $(x, y) \to (x, y)$, $(x, y) \to (x + 1, y)$, $(x, y) \to (x, y + 1)$, $(x, y) \to (x, x + y)$, and $(x, y) \to (-y, x)$) and considered the induced graph on Z_p^2. Gabber and Galil ([1], 1979) used Fourier analysis to prove that a very similar construction yields a family of expander graphs. They were also able to provide, for the first time, a lower bound on the expansion rate. This bound seems, however, far from being tight. Based on a theorem of Selberg (1965), Lubotzky, Phillips, and Sarnak ([2], 1986) showed that the Cayley graphs of $SL_2(Z_p)$ with respect to the generators $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ are expanders. This implies that the symmetric quotient graphs $Y_p = (V_p, E_p)$ defined by $V_p = Z_p \times Z_p \setminus \{(0, 0)\}$, $E_p = \{((a, b); (a, (b \pm a) \bmod p))\} \cup \{((a, b); ((a \pm b) \bmod p, b))\}$ are expanders (see also [3]). The celebrated LPS graphs are Ramanujan, i.e., they have the largest possible spectral gap. However, this fact yields only crude estimates for their expansion rate.

The main technical result in this paper is that for any planar compact set A of finite positive measure,

$$\frac{|S(A) \cup S^{-1}(A) \cup T(A) \cup T^{-1}(A) \cup A|}{|A|} \geqslant 2,$$

where $S(x, y) = (x, x + y)$ and $T(x, y) = (x + y, y)$.

We follow the great tradition of solving isoperimetric problems by means of symmetrization arguments. The vertex set of the graph we consider is the whole plane. We show that for any planar set A, we can find another set A' of the same area, that is "more symmetric" and expands at most as much as A does. We then determine the set of a given area that is invariant under the relevant symmetries and show that it is optimal. The proof is elementary.

Later we prove a similar statement for a directed version of the above graph. Namely,

$$\frac{|S(A) \cup T(A) \cup A|}{|A|} \geqslant \frac{4}{3},$$

for any planar compact set A of finite positive measure.