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A Riemann manifold structure of the spectra of weighted algebras of holomorphic functions

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To the memory of Goyo Sevilla, a good, honest man

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ABSTRACT

In this paper we give general conditions on a countable family V of weights on an unbounded open set U in a complex Banach space X such that the weighted space $HV(U)$ of holomorphic functions on U has a Fréchet algebra structure. For such weights it is shown that the spectrum of $HV(U)$ has a natural analytic manifold structure when X is a symmetrically regular Banach space, and in particular when $X = \mathbb{C}^n$.

1. Introduction

Weighted algebras of entire functions on $\mathbb{C}$ have been studied for many years. Berenstein, Li and Vidras in [1], Braun in [2] and Meise and Taylor in [3] used some particular weights to describe such algebras, for instance ones of infraexponential type. Motivated by this approach and by the very recent study by Carando and Sevilla-Peris [4] of weighted algebras of entire functions, essentially of exponential type, we study general conditions on a countable family V of weights such that the space $HV(U)$ has a Fréchet algebra structure. Moreover its spectrum $\mathfrak{M}V(U)$ has a natural analytic manifold structure whenever X is a symmetrically regular complex Banach space. This structure is based on the classical one given for any open (connected) subset of $\mathbb{C}^n$ (e.g. see [5, H.7. Lemma]), which was extended to the space $H_b(U)$ of holomorphic functions of bounded type on the open set U in [6] and to weighted algebras of entire functions, with weights of exponential type, in [4]. The notation and the approach of the proofs are infinitely dimensional, but the Riemann analytic structure obtained in Section 2 is new even for $\mathfrak{M}V(\mathbb{C}^n)$, $n = 1, 2, \ldots$, where our construction works since any finite dimensional Banach space is symmetrically regular. The main reason for writing the results in the setting of Banach spaces is that if X is a non-reflexive Banach space, then the Riemann structure on $\mathfrak{M}V(U)$ is obtained on the topological bidual of X. In the case of $U = X$, in Section 3, we prove (Theorem 3.7) that $\mathfrak{M}V(X)$ is a disjoint union of analytic copies of X^{**}. To do that, a key ingredient is extending the concept of associated weight given in [7] to the bidual of X.

By a Fréchet algebra we will understand an algebra for which the respective topological vector space is a Fréchet space in which the product is continuous (these are sometimes also called B_0-algebras). For a Banach space X, a function $P : X \to \mathbb{C}$

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is called an n-homogeneous polynomial if there exists a continuous n-linear mapping $L : X \times \cdots \times X \to \mathbb{C}$ such that $P(x) = L(x, \ldots, x)$ for every x. The space of all n-homogeneous polynomials is denoted by $\mathscr{P}(^nX)$. A function f is called holomorphic if for every point x there exists a $(P_n(x))_n$ (with each $P_n(x) \in \mathscr{P}(^nX)$) such that $f = \sum_n P_n(x)$ in some ball around x.

Also, for $x \in X$ and $r > 0$, $B_X(x, r)$ (resp. $\overline{B}_X(x, r)$) will denote the open (resp. the closed) ball centered at x with radius r. Given an open set $U \subseteq X$, by a weight we will understand any continuous function $v : U \to [0, \infty[$. Following [8,7,9–14,4,15–17] we consider a countable family $V = (v_n)_n$ of weights and define the space

$$HV(U) = \{f : U \to \mathbb{C}\text{: holom. } \|f\|_v = \sup_{x\in U} v(x)|f(x)| < \infty \text{ for all } v \in V\}.$$

It is worth mentioning that, since each $\|\cdot\|_v$ is a seminorm and the family V is countable, we are dealing with Fréchet spaces and (when that is the case) Fréchet algebras. Also, the fact that X is finite or infinite dimensional makes no difference in our study.

Given a weight v, the associated weight $\tilde{v}$ was defined in [7] by

$$\tilde{v}(x) = \frac{1}{\sup\{|f(x)| : f \text{ holomorphic}, \|f\|_v \le 1\}}.$$

It is well known that $v \le \tilde{v}$ [7, Proposition 1.2] and that, if U is absolutely convex, then $\|f\|_v = \|f\|_{\tilde{v}}$ for every f [7, Observation 1.12].

A set $A \subseteq U$ is said to be a U-bounded set if it is bounded and $d(A, X \setminus U) > 0$. The space of holomorphic functions on U that are bounded on U-bounded sets is denoted by $H_b(U)$. Following [15], we will say that a family of weights satisfies condition I if for every U-bounded set A there exists some $v \in V$ such that $\inf_{x\in A} v(x) > 0$. If condition I holds, then $HV(U)$ is continuously included in $H_b(U)$, a fact which we denote by $HV(U) \hookrightarrow H_b(U)$.

We will also consider the following conditions: for each $v \in V$ there exist $s > 0$, $w \in V$ and $C > 0$ such that

$$\operatorname{supp} v + \overline{B}_X(0, s) \subseteq U \tag{1}$$

$$v(x) \le C w(x + y) \quad \text{for all } x \in \operatorname{supp} v \text{ and all } y \in X \text{ with } \|y\| \le s. \tag{2}$$

We will say that a family of weights V has **good local control** if it satisfies condition I, conditions (1)–(2) and X^* is contained in $HV(U)$. Unless otherwise stated, we will always assume that V has good local control. Our interest in this good local control will become apparent in the next section.

2. The analytic structure of the spectrum

Given a Fréchet algebra $\mathscr{A}$, its spectrum is the set of all non-zero, continuous, linear and multiplicative functionals $\phi : \mathscr{A} \to \mathbb{C}$. Our aim in this section is to define an analytic structure on $\mathfrak{M}V(U)$, the spectrum of $HV(U)$. We will follow essentially the same trends as in [6] (see also [4] or [18, Section 6.3]).

It is known [4, Proposition 1] that $HV(U)$ is an algebra if and only if for every $v \in V$ there exist $w \in V$ and $C > 0$ such that $v(x) \le C\tilde{w}(x)^2$ for every $x \in U$. Clearly, this holds if we can get C and w such that

$$v(x) \le C w(x)^2. \tag{3}$$

The good local control will be crucial for the existence of the analytic structure on $\mathfrak{M}V(U)$ (see Theorem 2.12 and the lemmas preceding it). Let us then present some examples of families that enjoy this property jointly with (3), for which our main results in this section apply.

Example 2.1. In [15, Example 14] a family of weights V is defined such that $HV(U) = H_b(U)$. Obviously, this family V has good local control.

If we consider entire functions, condition (1) is trivially satisfied. In this case, a standard way to define a family of weights such that $HV(X)$ is an algebra is to take a continuous and decreasing function $\varphi : [0, \infty[\to]0, \infty[$ such that $\lim_{t\to\infty} t^k\varphi(t) = 0$ for every k (this condition is needed to get that $X^* \hookrightarrow HV(X)$) and then define weights $v_n(x) = \varphi(\|x\|)^{1/n}$. If we define the family $V = \{v_n\}_n$, condition (2) translates into restrictions on the decreasing rate of φ.

Proposition 2.2. *V defined as above has good local control if and only if there exist $\alpha \ge 1$ and $s > 0$ such that*

$$\sup_{t\in\mathbb{R}} \frac{\varphi(t)^\alpha}{\varphi(t+s)} < \infty. \tag{4}$$

Proof. Let us assume first that for each n there exist $s, C > 0$ and m such that $v_n(x) \le C v_m(x + y)$ for every x and $\|y\| \le s$ (i.e. V satisfies (2). Given $t \in \mathbb{R}$, let us choose $x \in X$ with $\|x\| = t$ and put $y = \frac{s}{\|x\|}x$. Then $\|x + y\| = t + s$ and we have $\varphi(t)^{1/n} \le C\varphi(t + s)^{1/m}$. Defining $\alpha = m/n$ we have (4).