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An application of Newton–Puiseux charts to the Jacobian problem

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ABSTRACT

We study 2-dimensional Jacobian maps using so-called Newton–Puiseux charts. These are multi-valued coordinates near divisors of resolutions of indeterminacies at infinity of the Jacobian map in the source space as well as in the target space. The map expressed in these charts takes a very simple form, which allows us to detect a series of new analytical and topological properties. We prove that the Jacobian Conjecture holds true for maps (f, g) whose topological degree is ≤ 5, for maps with $\gcd(\deg f, \deg g) \leq 16$ and for maps with. $\gcd(\deg f, \deg g)$ equal to 2 times a prime.

1. Introduction

A polynomial map $\mathbb{C}^2 \to \mathbb{C}^2$

$$(x, y) \to (f(x, y), g(x, y))$$

which satisfies the Jacobian equation

$$\mathrm{Jac}\,(f, g) = \begin{vmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{vmatrix} = 1$$

is called the Jacobian map. The Jacobian Conjecture states that any Jacobian map is invertible.

There are two major approaches to the 2-dimensional Jacobian Conjecture: algebraic and topological. In fact, these approaches are not that different; they represent different sides of a wider common approach.

The algebraic approach was developed mainly by Abhyankar [1]. It relies on expansions of the component polynomials f, g and of the Jacobian $\mathrm{Jac}(f, g)$ with respect to quasi-homogeneous gradations defined by means of edges of the Newton diagrams of the polynomials f, g. One obtains similarity of the Newton diagrams, reduction of their sizes and obstructions to the highest quasi-homogeneous parts of f and g. Further development in this direction was continued in the works of Moh [19], Appelgate and Onishi [2], Heitmann [14], Oka [25], Nagata [20,21], Nowicki and Nakai [24] and others. In particular, the Jacobian Conjecture was proved for maps with $\deg P \leq 101$, with $\gcd(\deg f, \deg g)$ being a prime number and with $\gcd(\deg f, \deg g) < 16$.

The topological approach was initiated by Vitushkin [31,32]. It essentially relies on resolution of indeterminacies of f and g at infinity. One obtains a holomorphic map between compact algebraic surfaces, which should be ramified (for a counter-example to the Jacobian Conjecture). In [31] an example of such a map is constructed in the topological category; it is 3–fold and is ramified along some divisor at 'infinity'. The paper [31] is very technical, but later Orevkov [29] explained Vitushkin's construction in terms of knots and fundamental groups (see also [34]). Recently, Egorov [10] has repeated Vitushkin's arguments in constructing a 5-fold map with two divisors of ramification.

Orevkov continued the study of the topology of Jacobian maps. In [28,30] he constructed examples of holomorphic maps from an open surface to $\mathbb{C}^2$ which are non-ramified outside a divisor with self-intersection $+1$ (like the line at infinity).

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Notations

f, g	polynomials
P	Jacobian map
$\hat{f}, \hat{g}, \widehat{P}$	corresponding rational maps
$\mathcal{X}^0 = \mathbb{C}^2$, $\mathcal{X} = \mathbb{CP}^2$, $\mathcal{X}_\infty$,	the source space, its closure and the line at infinity
$\mathcal{Y}^0 = \mathbb{C}^2$, $\mathcal{Y} = \mathbb{CP}^2$, $\mathcal{Y}_\infty$	the target space, its closure and the line at infinity
$\pi : \mathcal{Z} \to \mathcal{X}$	resolution of indeterminacies
$\theta, \tilde{\theta}$	N–P chart and N–P alteration chart
u, v	variables in $\theta, \tilde{\theta}$
γ_j, γ	exponents in θ
l_j, l, k	exponents in $\tilde{\theta}$
(r_j, k_j)	characteristic pairs for θ
$k^{(j)} = k_1 \dots k_j$	
a_j	coefficients in θ
$\tilde{f} = f \circ \theta$, $\tilde{g} = g \circ \theta$	
p, q	leading exponents on $\tilde{f}, \tilde{g}$
m, n	relatively prime and proportional to $\deg f$, $\deg g$
φ, ψ	leading coefficients in $\tilde{f}, \tilde{g}$
$\chi = \varphi^{1/m}$, $p_0 = p/m$	
$\Delta(f)$, $\Delta(f_d)$, $\Gamma(f)$, $\Gamma(f_d)$	Newton polygons and Newton diagrams
Δ_0, Γ_0	web Newton polygon and web Newton diagram
(α, β)	root vertex of Γ_0
I, I', J	edges in Newton diagrams
f_I	polynomial associated with I
$\mathfrak{A}, \mathfrak{S}, \mathfrak{N}$	intersection graph, splice diagram and Newton–Puiseux graph
$\Theta, \widetilde{\Theta}$	N–P charts in the target space
U, V	variables in $\Theta, \widetilde{\Theta}$
δ_j, δ	exponents in Θ and in quasi-Puiseux expansion
(ξ_j, η_j)	characteristic pairs for Θ
$\eta^{(j)} = \eta_0 \dots \eta_j$	
$b_j, c(u), \tilde{c}(z)$	coefficients in Θ and in quasi-Puiseux expansion
$\tilde{g}_N$	non-constant part of quasi-Puiseux expansion
$q_N, \tilde{\psi}_N$	leading exponent and leading coefficient in $\tilde{g}_N$
ϵ_i	exponents in φ
s	number of 'essential' factors in φ
$\nu = [(1-\gamma)\deg\varphi/p - 1]/k_d$	quantity in Proposition 4.2
$S(P), S_j$	the non-properness set and its components
$\mu_z(\cdot)$	multiplicity of a map at point z
μ_D	typical ramification index along divisor D
$\deg_{\mathrm{top}} D$	multiplicity of divisor D

He and A. Domrina proved that the Jacobian Conjecture holds true for maps of geometrical degree ≤ 4 [27,8,7]. Orevkov's approach uses so-called splice diagrams (introduced by Eisenbud and Neumann [11] in knot theory).

There was an attempt to prove the Jacobian Conjecture by showing topological equisingularity (or C^0-sufficiency) of the family of curves $\{f = \lambda\}$, see the works of Lê Dũng Tráng, Michel and Weber [16–18]. They also used diagrams to encode the resolution of singularities of the foliation $\{f = \lambda\}$.

It seems that the fundamental problem with the Jacobian Conjecture is the lack of a simple and effective formalism for the investigation of polynomial maps near infinity.

The aim of the present paper is to introduce new tools which unify the algebraic and topological approaches. We define the so-called Newton–Puiseux charts. In the source space the Newton–Puiseux charts are charts near some special divisors obtained after resolution of indeterminacies of P at infinity. On the other hand, they generalize the notion of weighted gradation associated with edges of the Newton diagrams of f and g. Their advantage relies on the simplicity of the Jacobian equation. Moreover, they are not overloaded with abstract formalism, they are only multi-valued changes of coordinates.

We use the Newton–Puiseux charts also in the target space. Then the representation of the Jacobian map leads to the so-called quasi-Puiseux expansion, which is an expansion of g in powers of f with many constant coefficients. The first non-constant coefficient is expressed via some Schwarz–Christoffel integral. Asymptotic analysis of these integrals yields certain