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On spineless cacti, Deligne's conjecture and Connes–Kreimer's Hopf algebra

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Abstract

Using a cell model for the little discs operad in terms of spineless cacti we give a minimal common topological operadic formalism for three a priori disparate algebraic structures: (1) a solution to Deligne's conjecture on the Hochschild complex, (2) the Hopf algebra of Connes and Kreimer, and (3) the string topology of Chas and Sullivan.

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0. Introduction

When considering an algebraic structure there is often a topological framework which is indicative of this structure. For instance, it is well known that Gerstenhaber algebras are governed by the homology operad of the little discs operad [2,3] and that Batalin–Vilkovisky algebras are exactly the algebras over the homology of the framed little discs operad [12]. The purpose of this paper is to prove that there is a common, minimal, topological operadic formalism [17,26] for three a priori disparate algebraic structures: (1) a homotopy Gerstenhaber structure on the chains of the Hochschild complex of an associative algebra, a.k.a. Deligne's conjecture, (2) the Hopf algebra of Connes and Kreimer [7], and (3) string topology [6].

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To accomplish this task, we use the spineless cacti operad of [17] which is responsible for the Gerstenhaber structure of string topology [4,6,17,19,26,40]. Moreover our operad of spineless cacti is actually equivalent to the little discs operad [17]. The analysis of this operad on the chain level allows us to give a new topological proof of Deligne's conjecture. Furthermore it provides chain models for the operads whose algebras are precisely pre-Lie algebras and graded pre-Lie algebras, respectively, as well as a chain realization for the Hopf algebra of Connes and Kreimer.

As we are using cacti, this approach naturally lies within string topology on one hand and on the other hand it is embedded in the framework of a combinatorial description of the moduli space of surfaces with punctured boundaries via the $\mathcal{A}rc$ operad [17,18,26,34]. Therefore all the previous structures obtain a representation in terms of moduli spaces.

We start by giving new CW decompositions for the spaces $\mathcal{C}act^1(n)$ of normalized spineless cacti with n lobes which are homotopy equivalent to the spaces $\mathcal{C}act(n)$ of spineless cacti with n lobes, the homotopy being the contraction of n factors of $\mathbb{R}_{>0}$.

Theorem 3.5. *The space $\mathcal{C}act^1(n)$ is homeomorphic to the CW complex $K(n)$.*

As shown in [17] the spaces $\mathcal{C}act^1(n)$ form a quasi-operad whose homology is an operad isomorphic to the homology operad of cacti and hence to the homology of the little discs operad. The operad structure, however, already appears on the chain level.

Theorem 3.11. *The glueings induced from the glueings of spineless normalized cacti make the spaces $CC_*(\mathcal{C}act^1(n))$ into a chain operad. Thus $CC_*(\mathcal{C}act^1)$ is an operadic model for the chains of the little discs operad.*

Moreover, the cells of $K(n)$ are indexed by planted planar bipartite trees and the operad of cellular chains is isomorphic to a combinatorial tree dg-operad. Reinterpreting the trees as "flow charts" for multiplications and brace operations and specifying appropriate signs, we obtain an operation of the cell operad of cacti and hence a cell model of the little discs operad on the Hochschild cochains of an associative algebra. This proves Deligne's conjecture in any characteristic, i.e. over $\mathbb{Z}$.

Theorem 4.3. *Deligne's conjecture is true for the chain model of the little discs operad provided by $CC_*(\mathcal{C}act^1)$, that is $CH^*(A, A)$ is a dg-algebra over $CC_*(\mathcal{C}act^1)$ lifting the Gerstenhaber algebra structure.*

Moreover, all possible flow charts using multiplication and brace operations are realized by the operations of the cells, and these operations are exactly the set of operations which appear when studying iterations of the bracket and the product on the Hochschild cochains. In this sense our solution to Deligne's conjecture is minimal.

Deligne's conjecture has by now been proven in various ways [1,23,24,29,30,35,39] (for a full review of the history see [32]). The different approaches are basically realized by choosing adequate chain models and some more or less abstract form of homological algebra. The virtue of our approach which is in spirit close to those of [29,24] lies in its naturality and directness. It yields a new topological proof, which is constructive, transparent and economical.

Restricting our attention to the suboperad of the operad of cellular chains of normalized spineless cacti given by symmetric top-dimensional cells $CC_n^{\mathrm{top}}(n)^{\mathbb{S}}$, we obtain a chain model for the operad $\mathcal{G}Pl$ whose algebras are precisely graded pre-Lie algebras. Suitably shifting degrees in this chain operad, we obtain the operad $\mathcal{P}l$ whose algebras precisely are pre-Lie algebras.