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A trace formula for the forcing relation of braids

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Abstract

The forcing relation of braids has been introduced for a 2-dimensional analogue of the Sharkovskii order on periods for maps of the interval. In this paper, by making use of the Nielsen fixed point theory and a representation of braid groups, we deduce a trace formula for the computation of the forcing order.

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Keywords: Braid groups; Braid forcing; Nielsen theory

1. Introduction

The influential statement "Period three implies chaos" of Li and Yorke [21] turns out to be a consequence of a much earlier theorem

Theorem 1.1 (*Sharkovskii [23]*). *On the set of natural numbers, define a linear order*

$$3 \succ 5 \succ 7 \succ \cdots \succ 2\cdot 3 \succ 2\cdot 5 \succ \cdots \succ 4\cdot 3 \succ 4\cdot 5 \succ \cdots \succ 8 \succ 4 \succ 2 \succ 1.$$

For any continuous map $f : [0, 1] \to [0, 1]$ of the interval, if f has a periodic orbit of period n, then f must have a periodic orbit of period m for every $m \prec n$.

However, in dimension 2, the same statement cannot be true, as shown by the $(2\pi/3)$-rotation of the unit disk. It was not until the work of Matsuoka [22] and Boyland [5], that the role of braids was revealed in the problem of the forcing relation of periodic orbits for homeomorphisms of the plane.

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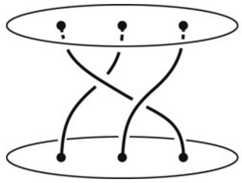

Fig. 1. A geometric braid.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation-preserving homeomorphism, and let $\{h_t : \mathbb{R}^2 \to \mathbb{R}^2\}_{0\le t\le 1}$ be an isotopy with $h_0 = \mathrm{id}$ and $h_1 = f$. An f-invariant set $P = \{x_1, \ldots, x_n\} \subset \mathbb{R}^2$ gives rise to a geometric braid (Ref. [3], see Fig. 1)

$$\{(h_t(x_i), t) \mid 0 \le t \le 1, 1 \le i \le n\}$$

in the cylinder $\mathbb{R}^2 \times [0, 1]$. Indeed, the closed curve $\{[h_t(x_1), \ldots, h_t(x_n)] \mid 0 \le t \le 1\}$ in the configuration space

$$\mathcal{X}_n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}^2, x_i \ne x_j, \forall i \ne j\}/\Sigma_n,$$

where Σ_n denotes the symmetric group of n symbols, gives rise to a braid β_P in the n-strand braid group $B_n = \pi_1(\mathcal{X}_n)$. With another connecting isotopy $\{h_t\}$, the resulting braid β_P may differ by a power of the “full-twist”. Matsuoka obtained lower bounds for the number of m-periodic points of $f|_{\mathbb{R}^2\setminus P}$, in terms of the trace of the reduced Burau representation of the braid $(\beta_P)^m$.

Later, Kolev [20] (see also [12]) found that a 3-periodic orbit P guarantees the existence of m-periodic orbits for every m, unless the braid β_P is conjugate to a power of the braid $\sigma_1\sigma_2$. Roughly speaking, this means that the $(2\pi/3)$-rotation mentioned above is the only exceptional case. Therefore, Li–Yorke’s statement still holds in a subtle way under 2-dimensional dynamics. The analogue of the Sharkovskii order naturally leads to the notion of forcing relation of (conjugacy classes of) braids.

In the following, the notation $[\beta]$ stands for the conjugacy class (in the group which is specified by the context) of a braid β.

Definition 1.2. A braid β *forces* a braid γ if, for any orientation-preserving homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ and any isotopy $\{h_t\} : \mathrm{id} \simeq f$, the existence of an f-invariant set P with $[\beta_P] = [\beta]$ guarantees the existence of an f-invariant set Q with $[\beta_Q] = [\gamma]$.

Remark 1.3. There is a homomorphism from the braid group B_n onto the mapping class group of the pair $(\mathbb{R}^2, P)$ (acting on $(\mathbb{R}^2, P)$ *from the right*), its kernel being generated by the “full-twist”. Via this homomorphism, $[\beta_P]$ is sent to the conjugacy class of the mapping class represented by f, which is independent of the choice of the isotopy $\{h_t\}$. Following Boyland [5], this invariant is referred to as the *braid type* of (f, P) in the literature. It is clear that the forcing relation of braids defined above naturally descends to that of braid types.

The forcing relation is essentially a problem concerning plane homeomorphisms. So the Bestvina–Handel theory of train-track maps [2] comes in naturally. By analyzing the symbolic dynamics of train-track maps, Handel [14] was able to totally solve the forcing relation among 3-strand pseudo-Anosov braids, and de Carvalho and Hall [7,8] have managed to do the same for horseshoe braids. This approach is, theoretically speaking, powerful enough to be extended to mapping classes of all punctured