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Mixed toric residues and tropical degenerations

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Abstract

Building on our earlier work on toric residues and reduction, we give a proof of the mixed toric residue conjecture of Batyrev and Materov. We simplify and streamline our technique of tropical degenerations, which allows one to interpolate between two localization principles: one appearing in the intersection theory of toric quotients and the other in the calculus of toric residues. This quickly leads to the proof of the conjecture, which gives a closed formula for the summation of a generating series whose coefficients represent a certain naive count of the numbers of rational curves on toric complete intersection Calabi–Yau manifolds.

Keywords: Toric varieties; Residues; Tropical geometry; Mirror symmetry

0. Introduction

This paper is a follow-up to our paper [16], where we prove a conjecture of Batyrev and Materov, the Toric Residue Mirror Conjecture (TRMC). Here we extend our results, and show that they imply a generalization of this conjecture, the Mixed Toric Residue Mirror Conjecture (MTRMC), which is also due to Batyrev and Materov [3].

Roughly, these conjectures state that the generating function of certain intersection numbers of a sequence of toric varieties converges to a rational function, which can be obtained as a finite residue sum on a single toric variety. We first recall the TRMC in some detail. We start with an integral convex polytope $\Pi^{\mathfrak{B}}$ in a d-dimensional real vector space $\mathfrak{t}$ endowed with a lattice of full rank $\mathfrak{t}_{\mathbb{Z}}$; we assume

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that the polytope contains the origin in its interior. Let the sequence $\mathfrak{B} = [\beta_1, \beta_2, \ldots, \beta_n]$ be the set of vertices of this polytope, ordered in an arbitrary fashion. One can associate a d-dimensional polarized toric variety $(V^{\mathfrak{B}}, L^{\mathfrak{B}})$ to this data in the standard fashion [16].

There is another way to obtain toric varieties from this data, which generalizes the mirror duality of polytopes introduced by Batyrev [1]. Consider the sequence $\mathfrak{A} = [\alpha_1, \alpha_2, \ldots, \alpha_n]$, which is the Gale dual of $\mathfrak{B}$ (cf. Section 1.3 for the construction). This is a sequence of integral vectors in the dual $\mathfrak{a}^*$ of a certain $r = n - d$-dimensional vector space $\mathfrak{a}$, which is also endowed with a lattice of full rank: $\mathfrak{a}_{\mathbb{Z}}$; in this setup the sequence $\mathfrak{A}$ spans a strictly convex cone $\mathrm{Cone}(\mathfrak{A})$. The simplicial cones generated by $\mathfrak{A}$ divide $\mathrm{Cone}(\mathfrak{A})$ into open chambers. Each chamber corresponds to a d-dimensional orbifold toric variety $V_{\mathfrak{A}}(\mathfrak{c})$ (cf. [8]). An integral element $\alpha \in \mathfrak{a}^*$ specifies an orbi-line-bundle L_α over this variety; denote the first Chern class of L_α by $\chi(\alpha) \in H^2(V_{\mathfrak{A}}(\mathfrak{c}), \mathbb{Q})$. For the purposes of this introduction we assume that this correspondence induces the linear isomorphisms

$$\mathfrak{a}^* \cong H^2(V_{\mathfrak{A}}(\mathfrak{c}), \mathbb{R}) \quad \text{and} \quad \mathfrak{a} \cong H_2(V_{\mathfrak{A}}(\mathfrak{c}), \mathbb{R}).$$

Now pick a chamber $\mathfrak{c}$ which contains the vector $\kappa = \sum_{i=1}^n \alpha_i$ in its closure: $\kappa \in \bar{\mathfrak{c}}$. To each element $\lambda \in \mathfrak{a}_{\mathbb{Z}}$, one can associate a moduli space MP_λ, the so-called Morrison–Plesser space (cf. [12]), which is a compactification of the space of those maps from the projective line to the variety $V_{\mathfrak{A}}(\mathfrak{c})$ under which the image of the fundamental class is λ:

$$\{m : \mathbb{P}^1 \to V_{\mathfrak{A}}(\mathfrak{c});\, m_*([\mathbb{P}^1]) = \lambda\}.$$

The varieties MP_λ are toric, and such that, again, to each integral element $\alpha \in \mathfrak{a}^*$ one can associate a line bundle L_α on MP_λ; again we denote the corresponding Chern class in $H^2(\mathrm{MP}_\lambda)$ by $\chi(\alpha)$. The space MP_λ is defined to be empty, unless $\langle \alpha, \lambda \rangle \geqslant 0$ for every $\alpha \in \mathfrak{c}$. The set of vectors satisfying this condition forms a cone in $\mathfrak{a}$, which we denote by $\bar{\mathfrak{c}}^{\perp}$; this cone is called the *polar* cone of $\mathfrak{c}$.

The construction also provides a Poincare dual class

$$K_\lambda \in H^{2(\dim \mathrm{MP}_\lambda - d)}(\mathrm{MP}_\lambda, \mathbb{Q})$$

to the subspace of MP_λ of those maps which land in a generic zero-section Y of the line bundle L_κ. When $V_{\mathfrak{A}}(\mathfrak{c})$ is smooth, then Y is a Calabi–Yau manifold.

To probe the class K_λ, we fix a homogeneous polynomial $P(x_1, \ldots, x_n)$ of degree d in n variables, and consider the intersection numbers

$$\int_{\mathrm{MP}_\lambda} P(\chi(\alpha_1), \ldots, \chi(\alpha_n)) K_\lambda,$$

which are to be interpreted as analogs of numbers of rational curves in Y subject to certain conditions specified by the polynomial P.

Now let $z_1, \ldots, z_n \in \mathbb{C}^*$, and form the Laurent series

$$\sum_{\lambda \in \mathfrak{a}_{\mathbb{Z}}} \int_{\mathrm{MP}_\lambda} P(\chi(\alpha_1), \ldots, \chi(\alpha_n)) K_\lambda \prod_{i=1}^n z_i^{\langle \alpha_i, \lambda \rangle}. \tag{0.1}$$