Available online at www.sciencedirect.com
SCIENCE DIRECT®
Topology 45 (2006) 643–671
TOPOLOGY
www.elsevier.com/locate/top

Circle actions on simply connected 5-manifolds

János Kollár*

Princeton University, Princeton, NJ 08544-1000, USA

Received 23 May 2005

Abstract

The aim of this paper is to study compact 5-manifolds which admit fixed point free circle actions. The first result implies that the torsion in the second homology and the second Stiefel–Whitney class have to satisfy strong restrictions. We then show that for simply connected 5-manifolds these restrictions are necessary and sufficient.

Keywords: Circle action; 5-manifold; Seifert bundle; Sasakian geometry

It is easy to see that a simply connected compact 5-manifold L admits a free circle action iff $H_2(L, \mathbb{Z})$ is torsion free and the classification of free circle actions up to diffeomorphism is equivalent to the classification of simply connected compact 4-manifolds plus the action of their diffeomorphism group on the second cohomology (cf. [9, Proposition 10]).

Motivated by some questions that arose in connection with the study of complex analytic Seifert $\mathbb{C}^*$-bundles [14,16], this paper investigates compact 5-manifolds that admit circle actions where the stabilizer of every point is finite, that is, fixed point free circle actions. We show that in this case $H_2(L, \mathbb{Z})$ can contain torsion, but the torsion and the second Stiefel–Whitney class have to satisfy strong restrictions. We then show that for simply connected manifolds these restrictions are necessary and sufficient for the existence of a fixed point free circle action.

* Tel.: +1 085441000; fax: +1 6092581367.
E-mail address: kollar@math.princeton.edu.

0040-9383/$ - see front matter © 2006 Elsevier Ltd. All rights reserved.
doi:10.1016/j.top.2006.01.003

Definition 1. Let M be any manifold. Write its second homology as a direct sum of cyclic groups of prime power order

$$H_2(M,\mathbb{Z}) = \mathbb{Z}^k + \sum_{p,i} (\mathbb{Z}/p^i)^{c(p^i)} \quad \text{for some } k = k(M), \ \ c(p^i) = c(p^i, M). \tag{1.1}$$

The numbers k, $c(p^i)$ are determined by $H_2(M,\mathbb{Z})$ but the subgroups $(\mathbb{Z}/p^i)^{c(p^i)} \subset H_2(M,\mathbb{Z})$ are usually not unique. One can choose the decomposition (1.1) such that the second Stiefel–Whitney class map

$$w_2 : H_2(M,\mathbb{Z}) \to \mathbb{Z}/2$$

is zero on all but one summand $\mathbb{Z}/2^n$. This value n is unique and it is denoted by $i(M)$ [1]. This invariant can take up any value n for which $c(2^n) \neq 0$, besides 0 and ∞. Alternatively, $i(M)$ is the smallest n such that there is an $\alpha \in H_2(M,\mathbb{Z})$ such that $w_2(\alpha) \neq 0$ and α has order 2^n.

The existence of a fixed point free differentiable circle action puts strong restrictions on H_2 and on w_2.

Theorem 2. *Let L be a compact 5-manifold with $H_1(L,\mathbb{Z})=0$ which admits a fixed point free differentiable circle action. Then*:

(1) *For every prime p, we have at most $k+1$ nonzero $c(p^i)$ in* (1.1). *That is*, $\#\{i : c(p^i) > 0\} \leqslant k+1$.
(2) *One can arrange that $w_2 : H_2(L,\mathbb{Z}) \to \mathbb{Z}/2$ is the zero map on all but the $\mathbb{Z}^k + (\mathbb{Z}/2)^{c(2)}$ summands in* (1.1). *That is, $i(L) \in \{0, 1, \infty\}$.*
(3) *If $i(L) = \infty$ then $\#\{i : c(2^i) > 0\} \leqslant k$.*

These conditions are sufficient for simply connected manifolds:

Theorem 3. *Let L be a compact, simply connected 5-manifold. Then L admits a fixed point free differentiable circle action if and only if $w_2 : H_2(L,\mathbb{Z}) \to \mathbb{Z}/2$ satisfies the conditions* (2(1–3)).

The conditions are especially transparent for spin homology spheres.

Example 4. Let $c(p^i)$ be any sequence of even natural numbers, only finitely many nonzero. By [24], there is a unique simply connected, spin, compact 5-manifold L such that $H_2(L,\mathbb{Z}) \cong \sum_{p,i} (\mathbb{Z}/p^i)^{c(p^i)}$.

By Theorem 2, this L admits a fixed point free differentiable circle action iff for every prime p, at most one of the $c(p), c(p^2), c(p^3), \ldots$ is nonzero.

It should be noted that the proof does *not* give a classification of all fixed point free S^1-actions on any 5-manifold. In fact, we exhibit infinitely many topologically distinct fixed point free S^1-actions on every L as in Theorem 3. In principle the classification of all S^1-actions on 5-manifolds is reduced to a question on four-dimensional orbifolds, but the four-dimensional question is rather complicated.

5. The classification of fixed point free circle actions on 3-manifolds was considered by Seifert [23]. If M is a 3-manifold with a fixed point free circle action then the quotient space $F := M/S^1$ is a surface (without boundary in the orientable case). The classification of these *Seifert fibered* 3-manifolds $f : M \to F$ is thus equivalent to the classification of fixed point free circle actions. It should be noted