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Embedding infinite cyclic covers of knot spaces into 3-space

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Dedicated to the memory of Professor Shiing-shen Chern

Abstract

We say a knot k in the 3-sphere $\mathbb{S}^3$ has *Property IE* if the infinite cyclic cover of the knot exterior embeds into $\mathbb{S}^3$. Clearly all fibred knots have Property IE.

There are infinitely many non-fibred knots with Property IE and infinitely many non-fibred knots without property IE. Both kinds of examples are established here for the first time. Indeed we show that if a genus 1 non-fibred knot has Property IE, then its Alexander polynomial $\Delta_k(t)$ must be either 1 or $2t^2 - 5t + 2$, and we give two infinite families of non-fibred genus 1 knots with Property IE and having $\Delta_k(t) = 1$ and $2t^2 - 5t + 2$ respectively.

Hence among genus 1 non-fibred knots, no alternating knot has Property IE, and there is only one knot with Property IE up to ten crossings.

We also give an obstruction to embedding infinite cyclic covers of a compact 3-manifold into any compact 3-manifold.

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1. Introduction

In this paper all surfaces and 3-manifolds are orientable, and all surfaces in 3-manifolds are proper, embedded and two-sided. Suppose S (resp. P) is a surface (resp. 3-manifold) in a 3-manifold M, we use

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$M \setminus S$ (resp. $M \setminus P$) to denote the manifold obtained by cutting M along S (resp. removing int P, the interior of P, from M).

Suppose S is a connected non-separating surface in M. Then $X = M \setminus S$ has two copies of S in ∂X, denoted by $S^+ \sqcup S^-$. Taking countably many copies of X: $\{X_i\}_{i=-\infty}^{+\infty}$, and identifying S_{i-1}^+ with S_i^- for all i, we get an infinite cyclic cover of M, denoted by $\widetilde{M}_S$.

Let k be a knot in $\mathbb{S}^3$, $E(k)$ be the exterior of k, S be a Seifert surface of k. Then $E(k)$ has a unique infinite cyclic cover, simply denoted by $\widetilde{E}(k)$. If k is a fibred knot with fibre S, then $\widetilde{E}(k)$ is homeomorphic to $S \times \mathbb{R}$ which clearly embeds into $\mathbb{S}^3$. This paper will address the following

Question 1. *Suppose k is a non-fibred knot, when does $\widetilde{E}(k)$ embed into $\mathbb{S}^3$?*

The third named author was introduced to Question 1 during conversations with Professor Robert D. Edwards in the spring of 1984, and Edwards attributed Question 1 to Professor J. Stallings.

It is natural to ask the following more general and flexible

Question 2. *When does an infinite cyclic cover of a compact 3-manifold embed into a compact 3-manifold?*

Definition 1.1. We say a knot k in $\mathbb{S}^3$ has Property IE, if the infinite cyclic cover $\widetilde{E}(k)$ embeds into $\mathbb{S}^3$. We say a knot k in $\mathbb{S}^3$ has Property DIE, if $(\widetilde{E}(k), \tau) \subset (\mathbb{S}^3, f)$, that is, the deck transformation τ of $\widetilde{E}(k)$ embeds into a dynamical system f on $\mathbb{S}^3$. (We say a dynamical system g on a space P embeds into a dynamical system f on a space Y, denoted by $(P, g) \subset (Y, f)$, if there is an embedding $P \subset Y$ such that $f|P = g$.)

The organization of this paper is as below.

Sections 2 and 3 are the main parts of the paper. All knots involved in Sections 2 and 3 are of genus 1 and non-fibred. It is well known that the only genus 1 fibred knots are 3_1 and 4_1 in the knot table.

In Section 2, we give a partial positive answer to Questions 1 and 2. In Section 2.1, beginning with a discrete dynamical system f on $\mathbb{S}^3$ (or a compact 3-manifold Y), we construct a compact 3-manifold M (closed or with torus boundary) such that $(\widetilde{M}_S, \tau) \subset (\mathbb{S}^3, f)$ or $\subset (Y, f)$, where τ is the deck transformation on the infinite cyclic cover $\widetilde{M}_S$. In Section 2.2 we prove that the simplest non-trivial example provided by construction in Section 2.1 is $E(9_{46})$, the exterior of the 46-th knot of nine crossings in the knot table, see [11] or [3], therefore providing the first known positive example to Question 1. A subtle point in the verification is to choose a right projection of 9_{46}, which significantly simplifies the process. But a key point is to choose 9_{46} among all knots in $\mathbb{S}^3$ to compare with. In Section 2.3, we give a sufficient condition for the 3-manifolds constructed in Section 2.1 to be complements of knots in $\mathbb{S}^3$, and then we prove that there are infinitely many non-fibred genus 1 knots having Property DIE by invoking Thurston and Soma's results on Gromov volume of 3-manifolds.

In Section 3, we give a partial negative answer to Question 1. By invoking Freedman–Freedman's version of the Kneser–Haken finiteness theorem and results of Gabai (and Novikov) on foliation and on surgery, we prove that if a genus 1 non-fibred knot k has Property IE, then $E(k)$ is constructed as in Section 2.1, and hence k has Property DIE. It follows that the Alexander polynomial of such knots must be 1 or $2t^2 - 5t + 2$, and the Alexander invariant is also restricted. So "most" genus 1 non-fibred knots do not have Property IE. In particular, among all non-fibred genus 1 knots, no alternating knots have Property IE, and up to crossing numbers ≤ 10 only 9_{46} has Property IE. On the other hand, two infinite