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A Dichotomy Theorem and other results for a class of quotients of topological groups

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ABSTRACT

Topological properties of quotients of topological groups with respect to compact subgroups are investigated. It is observed that many classical results on topological groups can be extended to coset spaces of this kind. The main results of the paper are presented in the last sections. Among them is the Dichotomy Theorem for coset spaces with respect to a compact subgroup which says that every remainder of any such coset space is either pseudocompact or metric-friendly. Several metrizability conditions for coset spaces in terms of their remainders are given.

1. Introduction and preliminaries

We consider the following general situation. Suppose that G is a topological group and H is a closed subgroup of G. Then G/H stands for the quotient space of G which consists of left cosets xH, where $x \in G$. We call the spaces G/H so obtained *coset spaces*. The space G/H needn't be homeomorphic to a topological group, but it is always homogeneous and Tychonoff (a space X is called *homogeneous* if for each pair x, y of points in X there exists a homeomorphism h of X onto itself such that $h(x) = y$). The 2-dimensional Euclidean sphere S^2 is a coset space which is not homeomorphic to any topological group. This is a part of the folklore. There exists also a homogeneous compact Hausdorff space X such that X is not a coset space [15]. It is well-known that any compact topological group is a dyadic compactum, and what is especially interesting in this connection is that there exists a compact coset space which is not

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dyadic. For the sake of completeness, we recall how to identify a space of this kind. A space X is said to be *strongly locally homogeneous* if for each $x \in X$ and every open neighbourhood U of x, there exists an open neighbourhood V of x such that $V \subset U$ and, for every $z \in V$, there exists a homeomorphism h of X onto X such that $h(x) = z$ and $h(y) = y$, for each $y \in X \setminus V$. The next statement was established by R.L. Ford in [18]. See also [11, Proposition 3.5.12].

Proposition 1.1. *If a zero-dimensional T_1-space X is homogeneous, then it is strongly locally homogeneous.*

Using the above result, one can easily establish the next statement (see Theorem 3.5.15 in [11]): *every homogeneous zero-dimensional compact Hausdorff space X can be represented as a coset space of a topological group*. In particular, it follows that the two arrows compactum A_2 [14, 3.10.C] is a coset space. However, A_2 is also first-countable and non-metrizable. Therefore, A_2 is not dyadic. Note also that every first-countable topological group is metrizable. However, there exists a first-countable compact non-metrizable and even non-dyadic coset space – the space A_2.

In this article, coset spaces G/H and their remainders are considered under the compactness restriction on H. We call a coset space G/H *compactly-fibered* if H is compact. This study is motivated, in particular, by a sequence of results on remainders of topological groups obtained in [2–4], and more recently, in [8–10]. We start with some examples of theorems on topological groups which can be extended to certain coset spaces (see the next section). Then, we extend and improve the Dichotomy Theorem for remainders of topological groups obtained in [4]. According to this theorem, every remainder of a topological group is either pseudocompact or Lindelöf. The extended Dichotomy Theorem is applied to study the structure of remainders of coset spaces in compactifications. In particular, some metrizability theorems for such remainders are given in the last section.

A few open problems on coset spaces and their remainders are also formulated. Some further information on coset spaces can be found in the book [26] which treats these objects from the positions of the theory of uniform spaces giving a special attention to the behaviour of completeness properties in coset spaces. See also [17,22,27,28,21], and the rich list of references in the book [26].

Below "a space" always stands for "a Tychonoff topological space". A space X is said to be *paralindelöf* if every open cover of X can be refined by a locally countable open cover [12]. By a remainder of a space X we mean the subspace $bX \setminus X$ of a compactification bX of X. Recall that paracompact p-spaces are preimages of metrizable spaces under perfect mappings. A mapping is said to be perfect if it is continuous, closed, and all fibers are compact. A Lindelöf p-space is the preimage of a separable metrizable space under a perfect mapping. K. Nagami has defined [24] the important class of Σ-spaces which contains the class of metrizable spaces and the class of spaces with a σ-discrete network. Lindelöf Σ-spaces can be characterized as continuous images of Lindelöf p-spaces. In particular, every Lindelöf p-space is in this class. Every space with a countable network, and every σ-compact space is in the class of Lindelöf Σ-spaces as well. We follow notation and terminology in [14]. A space X is of point-countable type if each $x \in X$ is contained in a compact subspace F of X with a countable base of open neighbourhoods in X.

2. Extensions to coset spaces of some theorems on topological groups

M.M. Choban established the following deep result on compact G_δ-subsets of topological groups: every space of this kind is dyadic [13]. Let us assume now that $X = G/H$ is a coset space where the subgroup H is compact, and let F be a compact G_δ-subset of X. The natural mapping g of G onto $X = G/H$ is perfect, since H is compact (see [11, Theorem 1.5.7]). Therefore, the preimage of F under g is a compact G_δ-subset P of G. Since G is a topological group, it follows, by Choban's Theorem, that P is a dyadic compactum. Since F is an image of P under the continuous mapping g, it follows that F is dyadic as well. Thus, the next generalization of Choban's Theorem holds: if F is a compact G_δ-subspace of a compactly-fibered coset