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Anti-Urysohn spaces [☆]

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ABSTRACT

All spaces are assumed to be infinite Hausdorff spaces. We call a space *anti-Urysohn* (*AU in short*) iff any two non-empty regular closed sets in it intersect. We prove that

- for every infinite cardinal κ there is a space of size κ in which fewer than $cf(\kappa)$ many non-empty regular closed sets always intersect;
- there is a locally countable AU space of size κ iff $\omega \leq \kappa \leq 2^{\mathfrak{c}}$.

A space with at least two non-isolated points is called *strongly anti-Urysohn* (*SAU in short*) iff any two infinite closed sets in it intersect. We prove that

- if X is any SAU space then $\mathfrak{s} \leq |X| \leq 2^{2^{\mathfrak{c}}}$;
- if $\mathfrak{r} = \mathfrak{c}$ then there is a separable, crowded, locally countable, SAU space of cardinality $\mathfrak{c}$;
- if $\lambda > \omega$ Cohen reals are added to any ground model then in the extension there are SAU spaces of size κ for all $\kappa \in [\omega_1, \lambda]$;
- if GCH holds and $\kappa \leq \lambda$ are uncountable regular cardinals then in some CCC generic extension we have $\mathfrak{s} = \kappa$, $\mathfrak{c} = \lambda$, and for every cardinal $\mu \in [\mathfrak{s}, \mathfrak{c}]$ there is an SAU space of cardinality μ.

The questions if SAU spaces exist in ZFC or if SAU spaces of cardinality $> \mathfrak{c}$ can exist remain open.

1. Introduction

In this paper "space" means "infinite Hausdorff topological space".

The space X is called *anti-Urysohn* (*AU*, in short) iff $A \cap B \neq \emptyset$ for any $A, B \in \mathrm{RC}^+(X)$, where $\mathrm{RC}^+(X)$ denotes the family of non-empty regular closed sets in X.

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We call the space X *strongly anti-Urysohn* (*SAU*, in short) iff $|X'| > 1$, that is X has at least two non-isolated points, and $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{F}^+(X)$, where $\mathcal{F}^+(X)$ denotes the family of *infinite* closed subsets of X. Clearly, AU spaces are crowded and a crowded SAU space is AU. Our original intention was to include crowdedness in the definition of SAU spaces. However, we changed our minds after we realized that it seems to be just as hard to construct them with the weaker property of having at least two non-isolated points. Actually, it is an easy consequence of our results given below that any SAU space has uncountably many non-isolated points.

What led us to consider AU spaces was not just idle curiosity. Co-operating via correspondence with Alan Dow, we have recently arrived at the result that in the Cohen model any separable and sequentially compact Urysohn space has cardinality $\leq \mathfrak{c}$. (This result will be published elsewhere.) The natural question if this holds for all (Hausdorff) spaces, however, remained open. When trying to find a ZFC counterexample, it was natural to look for spaces that are as much non-Urysohn as possible.

Actually, a countable AU space, under a different name, had been constructed by W. Gustin in [3] a long time ago, as a simple(r) example of a countable connected Hausdorff space. (It is obvious that AU spaces are connected.) The first example of a countable connected Hausdorff space was constructed by Urysohn in [7], but his construction is extremely long and complicated: just the description of his space takes up three pages. (We have no idea if Urysohn's example is AU or not.) A much simpler example was obtained by Gustin in [3] where the following was proved:

[3, Theorem 4.2] There is a countably infinite Hausdorff space X such that no two distinct points in X have disjoint closed neighborhoods (i.e. X is AU).

An even simpler construction of a countable connected Hausdorff space, which takes up only one page, was published by Bing in [1]. This is also presented as example 6.1.6 in Engelking's book [2]. Bing's example also turns out to be AU.

In contrast to this, we are not aware of any earlier appearance of SAU spaces. In fact, we admit that when we first considered them we did not think that they could exist.

Our notation and terminology is standard. In set theory we follow [6] and in topology [2].

2. Existence of anti-Urysohn spaces

In this section we show that for every infinite cardinal κ there is an AU space of cardinality κ. Actually, we prove much more that is new and of interest even for the case $\kappa = \omega$.

To do that we need the following somewhat technical lemma that provides a general method for constructing AU spaces.

Lemma 2.1. *Assume that κ is an infinite cardinal and X is a space with $X \cap \kappa = \emptyset$, moreover $\{K_\alpha : \alpha < \kappa\}$ are pairwise disjoint non-empty compact subsets of X such that*

(1) if $a \subset \kappa$ is cofinal then $\bigcup_{\alpha \in a} K_\alpha$ is dense in X;
(2) $Y = X \setminus \bigcup_{\alpha < \kappa} K_\alpha$ is also dense in X.

Define the topology ϱ on $Z = Y \cup \kappa$ as follows:

- *for $y \in Y$ the family $\{U \cap Y : y \in U \in \tau(X)\}$,*
- *for $\alpha \in \kappa$ the family*

$$\Big\{\{\alpha\} \cup (W \cap Y) : K_\alpha \subset W \in \tau(X)\Big\}$$

is a ϱ-neighborhood base. Then