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Approximation and interpolation by entire functions with restriction of the values of the derivatives ☆

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A R T I C L E I N F O

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Dedicated to Alan Dow on his 60th birthday

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A B S T R A C T

A theorem of Hoischen states that given a positive continuous function $\varepsilon : \mathbb{R}^n \to \mathbb{R}$, an unbounded sequence $0 \leq c_1 \leq c_2 \leq \dots$ and a closed discrete set $T \subseteq \mathbb{R}^n$, any C^∞ function $g : \mathbb{R}^n \to \mathbb{R}$ can be approximated by an entire function f so that for $k = 0, 1, 2, \dots$, for all $x \in \mathbb{R}^n$ such that $|x| \geq c_k$, and for each multi-index α such that $|\alpha| \leq k$,

(a) $|(D^\alpha f)(x) - (D^\alpha g)(x)| < \varepsilon(x)$;
(b) $(D^\alpha f)(x) = (D^\alpha g)(x)$ if $x \in T$.

In this paper, we show that if $C \subseteq \mathbb{R}^{n+1}$ is meager, $A \subseteq \mathbb{R}^n$ is countable and disjoint from T, and for each multi-index α and $p \in A$ we are given a countable dense set $A_{p,\alpha} \subseteq \mathbb{R}$, then we can require also that

(c) $(D^\alpha f)(p) \in A_{p,\alpha}$ for $p \in A$ and α any multi-index;
(d) if $x \notin T$, $q = (D^\alpha f)(x)$ and there are values of $p \in A$ arbitrarily close to x for which $q \in A_{p,\alpha}$, then there are values of $p \in A$ arbitrarily close to x for which $q = (D^\alpha f)(p)$;
(e) for each α, $\{x \in \mathbb{R}^n : (x, (D^\alpha f)(x)) \in C\}$ is meager in $\mathbb{R}^n$.

Clause (d) is a surjectivity property whose full statement in the text also allows for finding solutions in A to equations of the form $q = h^*(x, (D^\alpha f)(x))$ under similar assumptions, where $h(x, y) = (x, h^*(x, y))$ is one of countably many given fiber-preserving homeomorphisms of open subsets of $\mathbb{R}^{n+1} \cong \mathbb{R}^n \times \mathbb{R}$.
We also prove a weaker corresponding result with "meager" replaced by "Lebesgue null." In this context, the approximating function is C^∞ rather than entire, and we do not know whether it can be taken to be entire. The first result builds on earlier work of the author which deduced special cases of it from forcing theorems using absoluteness arguments. The proofs here do not use forcing.

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1. Introduction

A theorem of Carleman [8], extending the well-known theorem of Weierstrass on approximation by polynomials of continuous functions on compact intervals, states that for every continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and every positive continuous function $\varepsilon\colon \mathbb{R} \to \mathbb{R}$, there is a function $g\colon \mathbb{R} \to \mathbb{R}$ which is the restriction to $\mathbb{R}$ of an entire function and satisfies $|g(x) - f(x)| < \varepsilon(x)$ for all $x \in \mathbb{R}$. L. Hoischen proved the following generalization which allows approximation of both f and its derivatives. In this paper, t denotes a fixed positive integer.

Theorem 1.1 *([12], see also [11]). Let $g : \mathbb{R}^t \to \mathbb{C}$ and let N be a nonnegative integer. Let $\varepsilon : \mathbb{R}^t \to \mathbb{R}$ be a positive continuous function.*

(1) *If g is a C^N function, then there is an entire function $f\colon \mathbb{C}^t \to \mathbb{C}$ such that $|(D^\alpha f)(x) - (D^\alpha g)(x)| < \varepsilon(x)$ when $|\alpha| \leq N$.*
(2) *If g is a C^∞ function, then for each sequence $0 \leq c_0 \leq c_1 \leq \ldots$ of real numbers such that $\lim_{k\to\infty} c_k = \infty$, there is an entire function $f\colon \mathbb{C}^t \to \mathbb{C}$ such that for all $k = 0, 1, 2, \ldots$ $|(D^\alpha f)(x) - (D^\alpha g)(x)| < \varepsilon(x)$ when $x \in \mathbb{R}^t$, $|x| \geq c_k$ and $|\alpha| \leq k$.*

In both statements, if g takes real values on $\mathbb{R}^t$ then we may require the same property for f.

For the case $t = 1$, this theorem is improved in [13] to give approximation as well as interpolation on a closed discrete set. Hoischen's method can be adapted to functions of several variables.

Theorem 1.2 *([13], see also [6]). Let $g : \mathbb{R}^t \to \mathbb{C}$. Let $\varepsilon : \mathbb{R}^t \to \mathbb{R}$ be a positive continuous function. Let $T \subseteq \mathbb{R}^t$ be a closed discrete set.*

(1) *If g is a C^N function for some nonnegative integer N, then there exists an entire function f such that*
    (a) *$|(D^\alpha f)(x) - (D^\alpha g)(x)| < \varepsilon(x)$ for $x \in \mathbb{R}^t$, $|\alpha| \leq N$, and*
    (b) *$(D^\alpha f)(x) = (D^\alpha g)(x)$ for $x \in T$, $|\alpha| \leq N$.*
(2) *If g is a C^∞ function, then for each sequence $0 \leq c_0 \leq c_1 \leq \ldots$ of real numbers such that $\lim_{k\to\infty} c_k = \infty$, there exists an entire function f such that for all $k = 0, 1, 2, \ldots$*
    (a) *$|(D^\alpha f)(x) - (D^\alpha g)(x)| < \varepsilon(x)$ for $x \in \mathbb{R}^t$, $|x| \geq c_k$, $|\alpha| \leq k$, and*
    (b) *$(D^\alpha f)(x) = (D^\alpha g)(x)$ for $x \in T$, $|x| \geq c_k$, $|\alpha| \leq k$.*

In both statements, if g takes real values on $\mathbb{R}^t$ then we may require the same property for f.

This paper is mainly concerned with improvements to these theorems in which the approximating function f is also required to satisfy additional conditions either on the values of the derivatives $D^\alpha f$ at specific points or on the manner in which the graphs of f and its derivatives cut through a set $C \subseteq \mathbb{R}^{t+1}$ when C is either a meager set or a set of measure zero. We now describe our main results. The statements make use of the following notion.

Definition 1.3. A *fiber-preserving local homeomorphism* on $\mathbb{R}^{t+1} \cong \mathbb{R}^t \times \mathbb{R}$ is a homeomorphism $h\colon G_h^1 \to G_h^2$ between two open sets $G_h^1, G_h^2 \subseteq \mathbb{R}^{t+1}$ such that h has the form $h(x, y) = (x, h^*(x, y))$ for some continuous map $h^*\colon \mathbb{R}^{t+1} \to \mathbb{R}$. We write k_h for the inverse of h.

Remark 1.4. (i) k_h is also a fiber-preserving local homeomorphism.

(ii) $(x, y) \in G_h^1$ and $h^*(x, y) = z$ if and only if $(x, z) \in G_h^2$ and $k_h^*(x, z) = y$. In particular, the vertical sections of h^* are one-to-one: if $h^*(x, y_1) = h^*(x, y_2)$ then $y_1 = k_h^*(x, h^*(x, y_1)) = k_h^*(x, h^*(x, y_2)) = y_2$.