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Cofinality spectrum problems: The axiomatic approach ☆

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ABSTRACT

Our investigations are framed by two overlapping problems: finding the right axiomatic framework for so-called cofinality spectrum problems, and a 1985 question of Dow on the conjecturally nonempty (in ZFC) region of OK but not good ultrafilters. We define the lower-cofinality spectrum for a regular ultrafilter $\mathcal{D}$ on λ and show that this spectrum may consist of a strict initial segment of cardinals below λ and also that it may finitely alternate. We define so-called 'automorphic ultrafilters' and prove that the ultrafilters which are automorphic for some, equivalently every, unstable theory are precisely the good ultrafilters. We axiomatize a bare-bones framework called "lower cofinality spectrum problems", consisting essentially of a single tree projecting onto two linear orders. We prove existence of a lower cofinality function in this context and show by example that it holds of certain theories whose model theoretic complexity is bounded.

1. Introduction

Recall that two models M, N are elementarily equivalent, $M \equiv N$, if they satisfy the same sentences of first-order logic. A remarkable fact is that elementary equivalence may be characterized purely algebraically, without reference to logic:

Theorem A *(Keisler 1964 under GCH; Shelah unconditionally). $M \equiv N$ if and only if M, N have isomorphic ultrapowers, that is, if and only if there is a set I and an ultrafilter $\mathcal{D}$ on I such that $M^I/\mathcal{D} \cong N^I/\mathcal{D}$.*

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To prove this theorem, Keisler established that ultrafilters which are both regular and *good* exist on any infinite cardinal and that they have strong saturation properties. *For transparency in this introduction, all languages (e.g. vocabularies) are countable and all theories are (first-order) complete.* Regularity is an existential property of filters, showing a kind of strong incompleteness: a filter on I is regular if there exists a family $\mathcal{X} = \{X_i : i < |I|\} \subseteq \mathcal{D}$, called a regularizing family, such that the intersection of any infinitely many elements of $\mathcal{X}$ is empty. To motivate the definition of good, Definition 1.1 below, we finish outlining the proof of Theorem A in the GCH case. If $\mathcal{D}$ is regular and good, and $|M| \leq |I|$, then $M^I/\mathcal{D}$ is of size $2^{|I|}$ (since $\mathcal{D}$ is regular) and is $|I|^+$-saturated (since $\mathcal{D}$ is in addition good). So choosing $|I| \geq \max\{|M|, |N|\}$ and assuming the relevant instance of GCH, the ultrapowers $M^I/\mathcal{D}$, $N^I/\mathcal{D}$ are elementarily equivalent, of the same cardinality, and saturated in that cardinality, therefore isomorphic.

Given a model M and an ultrafilter $\mathcal{D}$, let us abbreviate '$M^I/\mathcal{D}$ is $|I|^+$-saturated' by writing '$\mathcal{D}$ saturates M'. In the proof just sketched, the saturation properties of good ultrafilters are tempered by regularity as follows. A good ultrafilter on I will saturate any M which is itself $|I|^+$-saturated (this may be taken as a definition of good ultrafilter, but see also Definition 1.1 below). If $\mathcal{D}$ is regular, then $\mathcal{D}$ saturates a model M if and only if it saturates all $N \equiv M$ (see Keisler [6] Theorem 2.1a). Since a good ultrafilter saturates some model in every elementary class (e.g. any one which is sufficiently saturated), a good regular ultrafilter saturates all models.

The usual definition of good filters is combinatorial. Call a function *monotonic* if $u \subseteq v$ implies $f(v) \subseteq f(u)$, and *multiplicative* if $f(u) \cap f(v) = f(u \cup v)$. In the following definition, it would suffice to consider all monotonic functions.

Definition 1.1 *(Good filters, Keisler).* Let $\mathcal{D}$ be a filter on I. We say $\mathcal{D}$ is κ-good if for every $\rho < \kappa$, every function $f : [\rho]^{<\aleph_0} \to \mathcal{D}$ has a multiplicative refinement, i.e. there is $g : [\rho]^{<\aleph_0} \to \mathcal{D}$ which is multiplicative and such that $g(u) \subseteq f(u)$ for all $u \in [\rho]^{<\aleph_0}$. We say $\mathcal{D}$ is *good* if it is $|I|^+$-good.

This has proved to be a very fruitful definition. The existence of good regular ultrafilters, proved by Keisler under GCH and by Kunen unconditionally, may be understood as asserting existence of ultrafilters which are 'maximal' or 'complex' in at least two senses: in the sense that all functions have multiplicative refinements, or in the sense of being strong enough to saturate any model. As a result, proposed weakenings of this definition have traditionally taken either a more set-theoretic form or a more model-theoretic form. An interesting example of the first is the notion of an 'OK' ultrafilter; see Dow 1985 [2], p. 146 for the history. Note that the cardinal parameter differs from Definition 1.1, i.e. a κ^+-good filter is κ-O.K.

Definition 1.2 *(OK filters).* Let $\mathcal{D}$ be a filter on I. We say $\mathcal{D}$ is κ-OK if every monotonic function $f : [\kappa]^{<\aleph_0} \to \mathcal{D}$ which satisfies $|u| = |v| \implies f(u) = f(v)$ has a multiplicative refinement. We say $\mathcal{D}$ is OK if it is $|I|$-OK.

It has been surprisingly difficult to distinguish OK from good. It follows from the existence of an $\aleph_1$-complete (non-principal) ultrafilter that there exist regular ultrafilters on any sufficiently large λ which are OK but not good (see for example Theorem 4.2 (4) $\nrightarrow$ (5) and Theorem 7.4 of [12]). We do not know of any ZFC proofs.

However, in his paper Dow raises a stronger question: "the question of whether there can be α^+-OK ultrafilters which are not α^+-good."

Question 1.3 *(Dow, cf. [2] 4.7).* Do there exist α^+-OK ultrafilters which are not α^+-good?

This question frames much of our present work. The theorem already quoted answers it assuming a measurable cardinal, and in fact allows for an arbitrary separation: