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Monotone covering properties and properties they imply

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Dedicated to Alan Dow, a fantastic mathematician who has made the second author smile uncountably many times

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ABSTRACT

We study properties of spaces that were proven in an earlier paper of Chase and Gruenhage (2013) [5] to follow from monotonic metacompactness. We show that all of the results of that earlier paper that follow from the monotonic covering property follow just from these weaker properties. The results we obtain are either strengthenings of earlier results or are new even for the monotonic covering property. In particular, some corollaries are that monotonically countably metacompact spaces are hereditarily metacompact, and separable monotonically countably metacompact spaces are metrizable. It follows that the well-known examples of stratifiable spaces given by McAuley and Ceder are not monotonically countably metacompact; we show that they are not monotonically meta-Lindelöf either. Finally, we answer a question of Gartside and Moody by exhibiting a stratifiable space which is monotonically paracompact in the locally finite sense, but not monotonically paracompact in the sense of Gartside and Moody.

1. Introduction

A space X is *monotonically (countably) metacompact [meta-Lindelöf]* if there is a function r that assigns to each (countable) open cover $\mathcal{U}$ of X a point-finite [point-countable] open refinement $r(\mathcal{U})$ covering X such that if $\mathcal{V}$ is an open cover of X and $\mathcal{V}$ refines $\mathcal{U}$, then $r(\mathcal{V})$ refines $r(\mathcal{U})$. The function r is called a *monotone (countable) metacompactness [meta-Lindelöfness] operator.*[1]

In [5] we introduced a couple of neighborhood assignment properties possessed by monotonically (countably) metacompact spaces. For a space X let $T(X)$ be the collection of all triples $p = (x^p, U_0^p, U_1^p)$ where U_0^p, U_1^p are open in X, and $x^p \in U_0^p \subset \overline{U_0^p} \subset U_1^p$.

Lemma 1.1. [5] *Let X be monotonically (countably) metacompact. Then to each $p \in T(X)$ one can assign an open V^p satisfying:*

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[1] Of course, every space is trivially "monotonically countably meta-Lindelöf" with $r(\mathcal{U}) = \mathcal{U}$.

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i *$x^p \in V^p \subset U_1^p$;*

ii *Whenever $\mathcal{Q} \subset T(X)$ ($\mathcal{Q}$ countable) then either $\bigcap_{q\in\mathcal{Q}} V^q = \emptyset$, or there exists a $\mathcal{Q}' \subset \mathcal{Q}$, with $\mathcal{Q}'$ finite, such that for any $q \in \mathcal{Q}$ there exists $q' \in \mathcal{Q}'$ such that either $V^q \subset U_1^{q'}$ or $V^q \cap U_0^{q'} = \emptyset$.*

Lemma 1.2. *[5] Let X be a monotonically (countably) metacompact T_3-space, and $Y \subset X$. For each $y \in Y$, if U_y is some open neighborhood containing y, then there exists an open neighborhood V_y of y with $V_y \subset U_y$ such that if $Y' \subset Y$ (Y' countable) and $\bigcap_{y\in Y'} V_y \neq \emptyset$, then there is a finite $Y'' \subset Y'$ such that $Y' \subset \bigcup_{y\in Y''} U_y$.*

These two lemmas were the main tools in proving the results of [5], and a similar lemma was the main tool of [7]. Thus it is natural to investigate just how strong the properties indicated by the conclusion of these lemmas are. Let (A) denote the property of the conclusion of Lemma 1.1, and (B) the conclusion of Lemma 1.2.[2] We thank an anonymous referee for pointing out simpler versions of (A) and (B) which will be formulated later. These simpler formulations are close in spirit to one of Borges' characterizations of monotone normality in terms of neighborhoods assigned to pairs (x, U) where U is an open neighborhood of x.

In this paper, we show that virtually all results of [5] following from a hypothesis of monotone (countable) metacompactness actually follow from (A), and some follow from (B). In particular, a compact Hausdorff space satisfying (A) must be metrizable. Furthermore, we prove that a space satisfying (B) is hereditarily metacompact, a fact which was not previously known about monotone metacompact spaces; the analogue for meta-Lindelöf also holds. We also prove that every separable space satisfying (A), hence every separable monotonically countably metacompact space, must be metrizable. The corollary that countable monotonically countably metacompact spaces are metrizable answers a question in [5].

Bennett, Hart, and Lutzer [3, Question 4.13] asked which stratifiable spaces are monotonically (countably) metacompact, and asked specifically about the well-known examples of stratifiable spaces due to Ceder and McAuley. We show that Ceder and McAuley's examples do not satisfy (A) or the meta-Lindelöf version of (A), hence are neither monotonically countably metacompact nor monotonically meta-Lindelöf. Finally, we answer a question of Gartside and Moody [6] (repeated by Stares [12]) by showing that a certain stratifiable space is monotonically paracompact in the locally finite sense, but not in the sense of Gartside and Moody. We end the paper with a question that indicates that there may be something fundamental yet to be proven about the structure of these classes of spaces.

All spaces are assumed to be regular and T_1.

2. Properties (A), (B), (C), and (D)

We first note simpler equivalent versions of the properties indicated by the conclusion of the lemmas stated in the introduction.

Definition 2.0. Let $P(X)$ be the collection of all pairs (x, U) where $x \in X$ and U is an open neighborhood of X. A space X has *property (A)* (resp., *property (B)*) if to each $(x, U) \in P(X)$, one can assign an open set $V(x, U)$ such that $x \in V(x, U) \subset U$ and such that for any collection $\{(x_\alpha, U_\alpha) : \alpha \in A\} \subset P(X)$, either $\bigcap_{\alpha\in A} V(x_\alpha, U_\alpha) = \emptyset$, or there exists a finite $A' \subset A$ such that for any $\alpha \in A$ there exists $\beta \in A'$ with $V(x_\alpha, U_\alpha) \subset U_\beta$ (resp., $\{x_\alpha : \alpha \in A\} \subset \bigcup_{\beta\in A'} U_\beta$).

[2] It would seem that there should be countable versions of (A) and (B), corresponding to monotonically countably metacompact, but as we shall see, the countable versions are equivalent to the unrestricted versions.