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On the problem of compact totally disconnected reflection of nonmetrizability ☆

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A R T I C L E I N F O

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To Alan Dow on his 60th birthday

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A B S T R A C T

We construct a ZFC example of a nonmetrizable compact space K such that every totally disconnected closed subspace $L \subseteq K$ is metrizable. In fact, the construction can be arranged so that every nonmetrizable compact subspace may be of fixed big dimension. Then we focus on the problem of whether a nonmetrizable compact space K must have a closed subspace with a nonmetrizable totally disconnected continuous image. This question has several links with the structure of the Banach space $C(K)$, for example, by Holsztyński's theorem, if K is a counterexample, then $C(K)$ contains no isometric copy of a nonseparable Banach space $C(L)$ for L totally disconnected. We show that in the literature there are diverse consistent counterexamples, most of them eliminated by Martin's axiom and the negation of the continuum hypothesis, but some consistent with it. We analyze the above problem for a particular class of spaces. OCA however, implies the nonexistence of any counterexample in this class but the existence of another absolute example remains open.

1. Introduction

This paper is related to the question whether a nonmetrizable space must have a nice, in some sense, nonmetrizable subspace. If the nice subspace that we seek means a subspace of small cardinality, positive consistent answers to this question were obtained by Alan Dow and others, for example, in [8,10,11,24,46]. When one restricts oneself to compact Hausdorff spaces the question if every nonmetrizable compact Hausdorff space has a nonmetrizable subspace of cardinality ω_1 has the positive answer in ZFC as proved by Alan Dow in [9]. Here we will ask about the reflection of the nonmetrizability for compact Hausdorff spaces to another type of nonmetrizable subspaces or quotient spaces, namely we want them to be totally disconnected and compact. Thus, the main questions are:

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Question 1.1. *Suppose that K is compact Hausdorff space which is nonmetrizable.*

(1) *Is there $L \subseteq K$ which is compact, nonmetrizable and totally disconnected?*
(2) *Is there a closed subspace $K' \subseteq K$ and a continuous surjective map $\phi : K' \to L$ such that L is nonmetrizable and totally disconnected?*

It is worthy to note that in Question 1.1 (2) (see [26], Question 4 (1176)) instead of continuous images of closed subspaces we could consider closed subspaces of continuous images (Lemma 4.1). Consistent examples providing positive answer to Question 1.1 (1) have been well known, for example, assuming the continuum hypothesis (CH), V. Fedorchuk showed that there are compact spaces where every infinite closed subspace has big dimension [15], assuming $\diamondsuit$ M. E. Rudin and P. Zenor constructed a nonmetrizable manifold where all closed subsets are metrizable or contain many copies of euclidean intervals ([39], 3.14 of [34]). It seems to be folkloric knowledge that the Souslin continuum is another example. Assuming $\clubsuit$ it is possible to construct a $\mathbb{T}$-bundle over the long ray like in Example 6.17 of [34] whose one point compactification provides another example. Some of these examples are consistent with any cardinal arithmetic, but some have continuous image, the compactification of the long ray, which contains a nonmetrizable totally disconnected subspace $[0, \omega_1]$. To obtain counterexamples to the second question from the above examples one needs to do a bit more work. We review these and other examples in the context of Question 1.1 (2) in Proposition 4.2.

In this note we focus especially on constructions of compact spaces of certain concrete type which do not need to be locally connected as many of the above examples. We call them split compact spaces in the analogy to the usual split interval (see e.g. [18]). Given a metrizable compact M, its points $\{r_\xi : \xi < \kappa\}$ for some cardinal κ and the splitting continuous functions $f_\xi : M \setminus \{r_\xi\} \to K_\xi$ where K_ξs are compact and metrizable we consider the split M induced by $(f_\xi)_{\xi<\kappa}$, for precise definition see 2.1. In particular, for us a split interval has a more general meaning than the usual split interval, to underline this difference we will talk about unordered split intervals in the nonclassical case. Such topological constructions can be traced back to Fedorchuk's school and found many applications in topology and in particular dimension theory (see [16]), and were rediscovered by Koppelberg in the context of totally disconnected spaces. Recently they and similar spaces have been applied in functional analysis in the connected version in [30] and totally disconnected version in [27,5].

The paper can be summarized as an attempt to construct spaces providing negative answers to Questions 1.1 (1) and (2) of the above form. Our main results are:

(a) *There is (in ZFC) a nonmetrizable compact Hausdorff space where every totally disconnected compact subspace is metrizable. Our example is an unordered split interval. (Theorem 3.2.)*
(b) *Assuming the existence of a Luzin set*[1] *there is an unordered split interval which is a nonmetrizable compact Hausdorff space without a continuous image containing a nonmetrizable totally disconnected closed subspace. (Theorem 4.3.)*
(c) *The existence of a compact space with no subspace with a continuous image which is a nonmetrizable and totally disconnected is consistent with Martin's axiom (MA) and the negation of CH. This is the Filippov split square, but split intervals or other examples can be arranged as well. (Theorem 4.5.)*
(d) *Assuming the Open Coloring Axiom*[2] *(OCA) every nonmetrizable split compact space has a continuous image with a nonmetrizable totally disconnected closed subspace. (Theorem 4.7.)*

[1] Recall that a Luzin set is an uncountable subset of the reals which meets every nowhere dense set only on a countable subset. Note that the assumption of the existence of a Luzin set is consistent with any cardinal arithmetics (just add ω_1 Cohen reals), it follows from CH and under the failure of CH it contradicts Martin's axiom.

[2] Recall that OCA (see 8.0 of [43]) developed by Abraham, Shelah and Todorcevic says that for any partition $[X]^2 = K_0 \cup K_1$ of a subset X of the reals such that K_0 is open in the product topology on $X \times X$ there is either an uncountable $Y \subseteq X$ such that $[Y]^2 \subseteq K_0$ or $X = \bigcup_{n\in\mathbb{N}} X_n$ where $[X_n]^2 \subseteq K_1$ for each $n \in \mathbb{N}$. OCA is consistent with MA, implies the failure of CH and follows from the Proper Forcing Axiom PFA or Martin's Maximum.