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Ramsey ultrafilters and Countable-to-one Uniformization

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ABSTRACT

We show that Countable-to-one Uniformization is preserved by forcing with $\mathcal{P}(\omega)/\mathrm{Fin}$ over a model of ZF in which every set of reals is completely Ramsey. We also give an exposition of Todorcevic's theorem that Ramsey ultrafilters are generic for $\mathcal{P}(\omega)/\mathrm{Fin}$ over suitable inner models.

1. Introduction

This paper presents a result on models of the form $M[U]$, where M is an inner model of ZF satisfying certain regularity properties inconsistent with the Axiom of Choice, and U is a Ramsey ultrafilter on the integers. Such extensions have been studied by several authors, notably Henle, Mathias and Woodin [6] and Di Prisco and Todorcevic [2,3], where the model M is variously taken to be a Solovay model or an inner model of Determinacy in the presence of large cardinals. Our result is that Countable-to-one Uniformization (a weak form of the Axiom of Choice; see the first paragraph of Section 3) is preserved by forcing with $\mathcal{P}(\omega)/\mathrm{Fin}$ over a model of ZF in which every set of reals is completely Ramsey (this includes many standard models of determinacy; see Section 3 and Subsection 1.2). In conjunction with the main result of [12], this fact can be used to show that there is no injection from $\mathcal{P}(\omega)/\mathrm{Fin}$ to $\mathbb{R}$ in models of the form the $M[U]$ considered here (a result previously proved in [3] by other means).

We let Fin denote the ideal of finite subsets of $\omega = \{0, 1, 2, \ldots\}$, and (for subsets x, y of ω) write $x \subseteq^* y$ for $x \setminus y \in \mathrm{Fin}$. It is easy to see that for any $\subseteq^*$-decreasing sequence $\langle x_n : n < \omega\rangle$ consisting of infinite

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subsets of ω, there is an infinite $y \subseteq \omega$ such that $y \subseteq^* x_n$ for all n. It follows that forcing with the Boolean algebra $\mathcal{P}(\omega)/\mathrm{Fin}$ over a model of $\mathrm{ZF} + \mathrm{DC}_{\mathbb{R}}$ does not add countable subsets of the ground model.[3] Forcing with this Boolean algebra over a model M of $\mathrm{ZF} + \mathrm{DC}_{\mathbb{R}}$ then produces a model $M[U]$, where the generic filter is naturally interpreted as a nonprincipal ultrafilter U on ω. In fact, the ultrafilter U is a selective (or Ramsey) ultrafilter, which means that for any collection $\{X_n : n \in \omega\} \subseteq U$ there is a set $\{i_n : n \in \omega\}$ (listed in increasing order) in U such that $i_0 \in X_0$ and each $i_{n+1} \in X_{i_n}$.

Ramsey ultrafilters exist if the Continuum Hypothesis holds, and their existence follows from weaker statements such as $cov(\mathcal{M}) = \mathfrak{c}$, where $cov(\mathcal{M})$ is the least cardinality of a collection of meager sets of reals whose union is the entire real line, and $\mathfrak{c}$ denotes the cardinality of the continuum (see Theorem 4.5.6 of [1]). Kunen [9] has shown that consistently there are no Ramsey ultrafilters.

A theorem of Todorcevic (see [4]) implies that in the context of large cardinals every Ramsey ultrafilter is generic over the inner model $L(\mathbb{R})$ for the partial order $\mathcal{P}(\omega)/\mathrm{Fin}$. We give a proof of this theorem in Section 2.

Woodin has shown that under the assumption of a proper class of Woodin cardinals, the theory of $L(\mathbb{R})$ is invariant under set forcing (see [11]). Since $\mathcal{P}(\omega)/\mathrm{Fin}$ is homogeneous, the theory of $L(\mathbb{R})[U]$ is also invariant under set forcing (in this context) when U is taken to be a Ramsey ultrafilter. This should mean that large cardinals give as detailed a theory for $L(\mathbb{R})[U]$ (i.e., answering most natural questions) as they do for the inner model $L(\mathbb{R})$. It remains to be seen whether this is the case. At the present moment many natural questions about this model remain open.

1.1. Notation

Given an infinite set $a \subseteq \omega$, we let $[a]^\omega$ denote the set of infinite subsets of a, and we let $[a]^{<\omega}$ denote the set of finite subsets of a (so $\mathrm{Fin} = [\omega]^{<\omega}$). Given $s \in [\omega]^{<\omega}$ and $a \in [\omega]^\omega$, we let $[s, a]$ denote the set of infinite subsets of $s \cup a$ with s as an initial segment. Given $s \in [\omega]^{<\omega}$ and a set $a \subseteq \omega$, a/s denotes a in the case that s is the emptyset, and $a \setminus (\max(s) + 1)$ otherwise.

1.2. Selective coideals and Ramsey ultrafilters

A *coideal* C on a set X is a subset of $\mathcal{P}(X)$ such that $\mathcal{P}(X) \setminus C$ is an ideal. Given $a \in C$, we let $C \restriction a$ denote $\{b \in C \mid b \subseteq a\}$. A coideal C on ω is *selective* if it contains all cofinite sets, and if for all $\subseteq$-decreasing sequences $\langle a_n : n \in \omega\rangle$ contained in C, there is a set $\{k_i : i \in \omega\}$ (listed in increasing order) in C such that $k_0 \in a_0$ and each k_{i+1} is in a_{k_i}. As defined above, a Ramsey ultrafilter is a selective ultrafilter on ω.

The following is part of Theorem 4.5.2 of [1].

Theorem 1.1. *A nonprincipal ultrafilter U on ω is Ramsey if and only if either of the following two statements holds.*

- *For every partition $\{y_n : n \in \omega\}$ of ω, either some $y_n \in U$ or there exists an $x \in U$ such that $|x \cap y_n| \leq 1$ for all $n \in \omega$.*
- *For all $a \subseteq [\omega]^2$, there is an $x \in U$ such that $[x]^2 \subseteq a$ or $[x]^2 \cap a = \emptyset$.*

Given $A \subseteq [\omega]^\omega$ and a coideal C on ω, we say that A is *C-Ramsey* (or has the *C-Ramsey property*) if there exists a $b \in C$ such that either $A \cap [b]^\omega = \emptyset$ or $[b]^\omega \subseteq A$. We say that A is *completely C-Ramsey* if for every finite $s \subseteq \omega$ and every $b \in C$, there exists a $d \in C \restriction b$ such that either $A \cap [s, d] = \emptyset$ or $[s, d] \subseteq A$. We drop the prefix C- when C is the coideal of infinite subsets of ω. It follows easily from the definitions

[3] The principle of Dependent Choices (DC) says that every tree of height ω without terminal nodes has an infinite branch; $\mathrm{DC}_{\mathbb{R}}$ is DC restricted to trees on $\mathbb{R}$.