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Some observations on the Baireness of $C_k(X)$ for a locally compact space X

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ABSTRACT

We prove some consistency results concerning the Moving Off Property for locally compact spaces, and thus the question of whether their function spaces are Baire.

1. Introduction

This paper is dedicated to Alan Dow, whom I have known for more than thirty years and in whose honor I defined *n-dowments* [5]. Lately, we have again been fruitfully collaborating, which means he has been solving problems I was interested in, but couldn't solve – a valuable service to humanity. See [3,4,2].

The Moving Off Property was introduced in [14] to characterize when $C_k(X)$ satisfies the Baire Category Theorem, for q-spaces X. Here we shall only be concerned with locally compact spaces (which are q), and so won't define q. We shall assume all spaces are Hausdorff.

Definition. A ***moving off*** *collection* for a space X is a collection $\mathcal{K}$ of non-empty compact sets such that for each compact L, there is a $K \in \mathcal{K}$ disjoint from L. A space satisfies the **Moving Off Property** (MOP) if each moving off collection includes an infinite subcollection with a discrete open expansion.

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Definition. $C_k(X)$, for a space X, is the collection of all continuous real-valued functions on X, considered as a subspace of the compact-open topology on the Cartesian power $X^{\mathbb{R}}$.

Theorem 1 *([14]). A locally compact space X satisfies the MOP if and only if $C_k(X)$ is Baire, i.e. satisfies the Baire Category Theorem.*

There is a more elementary equivalent of the MOP for locally compact spaces:

Lemma 2 *([13]). Let X be a locally compact space. Then X has the MOP if and only if every moving off collection for X includes an infinite discrete subcollection.*

Proof. Let $\mathcal{K}$ be a moving off collection for X. By local compactness, each $K \in \mathcal{K}$ can be fattened to an open set with compact closure. Let $\mathcal{K}'$ be the collection of all compact closures of open sets around members of $\mathcal{K}$. Then $\mathcal{K}'$ is moving off. Let C be a compact subset of X. There is a $K \in \mathcal{K}$ disjoint from C. By regularity and local compactness, there is an open $U \supseteq K$ with compact closure $\overline{U}$ disjoint from C. Then $\overline{U} \in \mathcal{K}'$. Since we have established that $\mathcal{K}'$ is moving off, by hypothesis it includes an infinite discrete collection $\{\overline{U}_n\}_{n<\omega}$. But each U_n included some $K_n \in \mathcal{K}$. Then $\{K_n\}_{n<\omega}$ is discrete and has the discrete open expansion $\{U_n\}_{n<\omega}$. □

We shall be referring to a couple of models of set theory not as well known as the usual ones. We will not need to know their inner workings, but just their properties established elsewhere. We briefly describe them:

Model I. Start with a supercompact cardinal. Force to make it indestructible under countably closed forcing. Force to establish $\diamondsuit$ for stationary systems at all regular cardinals. Then there is a coherent Souslin tree S. Force PFA(S), i.e. PFA restricted to partial orders that preserve S. Then force with S.

Model II. This is a cut-down version of Model I, designed to avoid large cardinals. We eliminate the supercompact cardinal and the first forcing. Rather than forcing PFA(S), we force several consequences of it which do not require large cardinals, before forcing with S. (Those consequences are: $\sum^-$ for compact sequential spaces [7]; countably compact, non-compact, first countable spaces include copies of ω_1 [4].) This model is introduced in [4], which depends on [3]. The reader who (justifiably!) does not want to rely on not yet available papers should replace Model II by Model I everywhere, and assume the supercompact cardinal.

In other papers, I have used the notation *PFA(S)[S] implies* Φ as a shorthand for "start with a coherent Souslin tree S. Force PFA(S). Then force S. Φ holds in the resulting model." I have used Φ *holds in a model of form PFA(S)[S]* to mean the process quoted above, possibly with additional steps. In particular, Model I is a model of form PFA(S)[S].

Lemma 3 *([17,4]). In Models I and II, locally compact, perfectly normal spaces are paracompact.*

Let us also quote several useful results concerning the MOP.

Lemma 4 *([19,14]). Countably compact spaces satisfying the MOP are compact.*

Lemma 5 *([19,14]). First countable spaces satisfying the MOP are locally compact.*

Lemma 6 *([19,14]). Locally compact, paracompact spaces satisfy the MOP.*

A stronger result is in Lemma 17 below.