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Structure theorems for finite unions of subspaces of special kind ☆

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To Alan Dow whose works reveal the vibrant beauty of Topology with astounding force!

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ABSTRACT

We study the internal structure of topological spaces X which can be represented as the union of a finite collection of subspaces belonging to some nice class of spaces. Several closely related structure theorems are established. In particular, they concern the finite unions of subspaces with the weight $\leq \tau$, the finite unions of subspaces with a point-countable base, and the finite unions of metrizable subspaces. As a corollary, we extend to finite unions the classical Mischenko's Theorem on metrizability of compacta with a point-countable base [11] (see Theorem 11). A few other applications of the structure theorems are given, in particular, to homogeneous spaces (Corollaries 5 and 10).

1. Introduction

In this paper, we present several versions of a structure theorem for topological spaces which can be represented as the union of a finite, or almost finite, collection of subspaces of certain kind. Special attention is given to finite unions of metrizable subspaces and to the more general case of finite unions of subspaces with a point-countable base. Observe that some deep results on compacta and countably compact spaces which can be represented as the union of a countable collection of metrizable subspaces, or of subspaces which resemble metrizable subspaces in many respects, were obtained by A.J. Ostaszewskii [13] and M.G. Tkachenko [14].

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In particular, we consider below an arbitrary countably compact space which is the union of a finite collection γ of subspaces with a point-countable base, and show that every space of this kind is the union of the same number or less of metrizable subspaces. Note that the original subspaces with a point-countable base need not be metrizable. The case of finite unions of subspaces with a point-countable base is principally deeper than the case of finite unions of metrizable subspaces (see Theorem 11). Indeed, it is almost trivial to show that every countably compact metrizable space is separable, but a similar fact for countably compact spaces with a point-countable base is a deep result of A.S. Mischenko [11]. Some further applications of the structure theorems are given.

Let us describe the basic construction which is present in the proofs of all structure theorems below. Suppose that $\gamma = \{M_i : i = 1, \dots, n\}$ is a finite (indexed) family of subsets of a space X and n is the number of elements in γ (counted indexwise). Put $H_i = \overline{M_i}$, for every $M_i \in \gamma$, and $\xi = \{H_i : i = 1, \dots, n\}$. A subfamily μ of ξ will be called *centered* if $\cap\mu$ is nonempty, and *maximal centered* if μ is centered and is not contained in any larger (indexwise) centered subfamily of ξ. The maximum of numbers of elements in centered subfamilies of ξ (counted indexwise) will be called the *ci-index*, or the *closure-intersection index*, of the system (X, γ), and is denoted by $ci(X, \gamma)$. Let $\S$ be the family of all maximal centered subfamilies of ξ, and $\eta = \{\cap\mu : \mu \in \S\}$. We will call the family η the *closure-derivative* of γ, and the subspace $(X)_m = \cup\eta$ of X will be called the *kernel* of γ (in X). Clearly, $ci(X, \gamma) \leq n$. If members of γ are not assumed to be subsets of X, then we put $ci(X, \gamma) = ci(X, \gamma_X)$, where $\gamma_X = \{M \cap X : M \in \gamma\}$. Notice that the empty members of γ can be ignored. We use throughout the article the terminology and notation we have just introduced.

"A space" stands below for a regular topological T_1-space. For standard concepts we use the terminology and notation from [7]. The weight of a space X is denoted by $w(X)$. Recall that $w(X)$ is the smallest infinite cardinal number τ such that there exists a base $\mathcal{B}$ for X with $|\mathcal{B}| \leq \tau$. A *semi-open* subset of a space is a set which can be represented as the intersection of an open set with a closed set. A *semi-open decomposition* of a space X is a disjoint covering of X by semi-open sets.

2. The structure theorem for finite unions of spaces with the weight $\leq \tau$

Theorem 1. *Suppose that τ is an infinite cardinal number, and X is a space which is the union of a finite family γ of subspaces such that $w(Y) \leq \tau$ for each $Y \in \gamma$. Then X can be represented as follows: $X = \cup\{F_i : i = 1, \dots, n\}$, where each F_i is closed in X, $F_i \subset F_{i+1}$ for $i = 1, \dots, n-1$, $w(F_1) \leq \tau$, $w(F_2 \setminus F_1) \leq \tau$, ..., $w(F_n \setminus F_{n-1}) \leq \tau$, and n doesn't exceed the cardinality of γ.*

In the proof of the above theorem we assume that $\tau = \omega$. The proof for an arbitrary infinite cardinal τ is practically the same. The key step in the argument is the next result:

Proposition 1. *Suppose that a space X is the union of a finite family $\gamma = \{M_i : i = 1, \dots, n\}$ of subspaces with a countable base. Let $H_i = \overline{M_i}$ for $i \in \{1, \dots, n\}$, $\xi = \{H_i : i = 1, \dots, n\}$, and let $\S$ be the family of all maximal centered subfamilies of ξ, $\eta = \{\cap\mu : \mu \in \S\}$, $(X)_m = \cup\eta$, $(X)_c = X \setminus (X)_m$, $M_{ic} = M_i \setminus (X)_m$ for $i = 1, \dots, n$.*

Then η is a disjoint family of closed subsets of X, $(X)_m = \cup\eta$ is a closed subspace of X with a countable base, and the ci-index of the system $((X)_c, \{M_{ic} : i = 1, \dots, n\})$ is smaller than the ci-index of $(X, \{M_i : i = 1, \dots, n\})$ (which doesn't exceed n).

Proof. Clearly, all members of η are closed in X. If $\mu_1, \mu_2 \in \S$ are such that the sets $P_1 = \cap\mu_1$ and $P_2 = \cap\mu_2$ are distinct but not disjoint, then $\mu = \mu_1 \cup \mu_2$ is a centered subfamily of ξ, and $\mu_1 \subset \mu$, $\mu_2 \subset \mu$. Then, by maximality, $\mu_1 = \mu = \mu_2$, a contradiction. Therefore, η is disjoint.

Claim 0: Each $P \in \eta$ is separable and metrizable.