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# The Katowice problem and autohomeomorphisms of $\omega_0^*$

David Chodounský [a,1], Alan Dow [b,2], Klaas Pieter Hart [c,*], Harm de Vries [d]

[a] *Institute of Mathematics, AS CR, Žitná 25, 115 67 Praha I, Czech Republic*
[b] *Department of Mathematics, UNC-Charlotte, 9201 University City Blvd., Charlotte, NC 28223-0001, United States*
[c] *Faculty of Electrical Engineering, Mathematics and Computer Science, TU Delft, Postbus 5031, 2600 GA Delft, The Netherlands*
[d] *Levendaal 143-A, 2311 JH Leiden, The Netherlands*

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The other authors dedicate this paper to Alan, who doesn't look a year over 59

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A B S T R A C T

We show that the existence of a homeomorphism between $\omega_0^*$ and $\omega_1^*$ entails the existence of a non-trivial autohomeomorphism of $\omega_0^*$. This answers Problem 441 in [8].

We also discuss the joint consistency of various consequences of $\omega_0^*$ and $\omega_1^*$ being homeomorphic.



* Corresponding author.
*E-mail addresses:* david.chodounsky@matfyz.cz (D. Chodounský), adow@uncc.edu (A. Dow), k.p.hart@tudelft.nl (K.P. Hart), vries101@gmail.com (H. de Vries).
*URLs:* http://www.math.uncc.edu/~adow (A. Dow), http://fa.its.tudelft.nl/~hart (K.P. Hart).

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## 0. Introduction

The Katowice problem, posed by Marian Turzański, is about Čech–Stone remainders of discrete spaces. Let $\kappa$ and $\lambda$ be two infinite cardinals, endowed with the discrete topology. The Katowice problem asks:

If the remainders $\kappa^*$ and $\lambda^*$ are homeomorphic must the cardinals $\kappa$ and $\lambda$ be equal?

Since the weight of $\kappa^*$ is equal to $2^\kappa$ it is immediate that the Generalized Continuum Hypothesis implies a yes answer. In joint work Balcar and Frankiewicz established that the answer is actually positive without any additional assumptions, *except possibly for the first two infinite cardinals*. More precisely

**Theorem** *([1,5]). If $\langle\kappa,\lambda\rangle \neq \langle\aleph_0,\aleph_1\rangle$ and $\kappa < \lambda$ then the remainders $\kappa^*$ and $\lambda^*$ are not homeomorphic.*

This leaves open the following problem.

**Question.** Is it consistent that $\omega_0^*$ and $\omega_1^*$ are homeomorphic?

Through the years various consequences of "$\omega_0^*$ and $\omega_1^*$ are homeomorphic" were collected, in the hope that their conjunction would imply $0 = 1$ and thus yield a full positive answer to the Katowice problem.

In the present paper we add another consequence, namely that there is a non-trivial autohomeomorphism of $\omega_0^*$. Whether this is a consequence was asked by Nyikos in [7] (as Problem 441 in the whole volume [8]), right after he mentioned the relatively easy fact that $\omega_1^*$ has a non-trivial autohomeomorphism if $\omega_0^*$ and $\omega_1^*$ are homeomorphic, see the end of Section 1.

After some preliminaries in Section 1 we construct our non-trivial autohomeomorphism of $\omega_0^*$ in Section 2. In Section 3 we shall discuss the consequences alluded to above and formulate a structural question related to them; Section 4 contains some consistency results regarding that structural question.

## 1. Preliminaries

We deal with Čech–Stone compactifications of discrete spaces exclusively. Probably the most direct way of defining $\beta\kappa$, for a cardinal $\kappa$ with the discrete topology, is as the space of ultrafilters of the Boolean algebra $\mathcal{P}(\kappa)$, as explained in [6] for example.

The remainder $\beta\kappa \setminus \kappa$ is denoted $\kappa^*$ and we extend the notation $A^*$ to denote $\operatorname{cl} A \cap \kappa^*$ for all subsets of $\kappa$. It is well known that $\{A^* : A \subseteq \kappa\}$ is exactly the family of clopen subsets of $\kappa^*$.

All relations between sets of the form $A^*$ translate back to the original sets by adding the modifier "modulo finite sets". Thus, $A^* = \emptyset$ iff $A$ is finite, $A^* \subseteq B^*$ iff $A \setminus B$ is finite and so on.

This means that we can also look at our question as an algebraic problem:

**Question.** Is it consistent that the Boolean algebras $\mathcal{P}(\omega_0)/\mathit{fin}$ and $\mathcal{P}(\omega_1)/\mathit{fin}$ are isomorphic?

Here *fin* denotes the ideal of finite sets. Indeed, the algebraically inclined reader can interpret $A^*$ as the equivalence class of $A$ in the quotient algebra and read the proof in Section 2 below as establishing that there is a non-trivial automorphism of the Boolean algebra $\mathcal{P}(\omega_0)/\mathit{fin}$.

### 1.1. Auto(homeo)morphisms

It is straightforward to define autohomeomorphisms of spaces of the form $\kappa^*$: take a bijection $\sigma : \kappa \to \kappa$ and let it act in the obvious way on the set of ultrafilters to get an autohomeomorphism of $\beta\kappa$ that leaves