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Contractibility is not a strong Whitney-reversible property for locally connected continua ☆

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We show an example of a noncontractible locally connected continuum X that admits a Whitney map μ for $C(X)$ such that $\mu^{-1}(t)$ is an absolute retract for each $t \in (0, 1)$.

1. Introduction

A separable metric space X is called an *absolute retract* (abbreviated AR) provided that X is a retract of every separable metric space containing X as a closed subspace. A *continuum* is a compact connected metric space with more than one point. Given a continuum X, we consider its hyperspaces:

$$2^X = \{A \subset X : A \text{ is closed and nonempty}\},$$

and

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$$C(X) = \{A \in 2^X : A \text{ is connected}\}.$$

These hyperspaces are endowed with the Hausdorff metric H.

Let X be a continuum. A *Whitney map* for $C(X)$ is a continuous function $\mu : C(X) \to [0, \infty)$ such that

(1) $\mu(\{x\}) = 0$ for each $x \in X$ and
(2) if $A, B \in C(X)$ and $A \subsetneq B$, then $\mu(A) < \mu(B)$.

If μ is a Whitney map for $C(X)$ and $t \in [0, \mu(X))$, then $\mu^{-1}(t)$ is called a *Whitney level*.

A topological property P is said to be a *strong Whitney-reversible property* provided that whenever X is a continuum such that $\mu^{-1}(t)$ has property P for some Whitney map μ for $C(X)$ and all $t \in (0, \mu(X))$, then X has property P.

In [2, Remarks 4.5], Jack T. Goodykoontz, Jr. and Sam B. Nadler, Jr. asked the following question (the same question appears in [3, Question 41.9]).

Question 1.1. *Is the property of being contractible a strong Whitney-reversible property for the class of Peano continua?*

Our purpose in this paper is to answer Question 1.1 negatively, *i.e.*, we give an example of a noncontractible locally connected continuum X that admits a Whitney map μ for $C(X)$ such that $\mu^{-1}(t)$ is an AR, so it is contractible, for each $t \in (0, \mu(X))$.

2. Preliminary results

The *cone* over a topological space Z is denoted by *cone*(Z). A *free arc* in a continuum X is an arc A contained in X with endpoints p and q and satisfying that $A \setminus \{p, q\}$ is open in X.

Let X be a continuum. A Whitney map μ for $C(X)$ is called an *admissible Whitney map for* $C(X)$ provided that there is a continuous homotopy $H : C(X) \times [0, 1] \to C(X)$ satisfying the following two conditions:

(i) for all $A \in C(X)$, $H(A, 1) = A$ and $H(A, 0) \in F_1(X)$;
(ii) if $\mu(H(A, t) > 0$ for some $A \in C(X)$ and $t \in [0, 1]$, then $\mu(H(A, s)) < \mu(H(A, t))$ whenever $0 \le s < t$.

In [2], Jack T. Goodykoontz, Jr. and Sam B. Nadler, Jr. made a detailed study of admissible Whitney maps, in particular they proved the following two results:

Theorem 2.1. [2, 2.15] *If X is the cone over any nonempty compact metric space Y, then there is an admissible Whitney map for $C(X)$.*

Theorem 2.2. [2, 4.1] *Let X be a locally connected continuum. If there is an admissible Whitney map μ for $C(X)$ and X contains no free arc, then $\mu^{-1}(t_0)$ is a Hilbert cube whenever $0 < t_0 < \mu(X)$.*

Given a continuum X and μ a Whitney map for $C(X)$, for $p \in X$, let $C_p(X) = \{A \in C(X) : p \in X\}$ be the relative hyperspace and let μ_p denote the restriction of μ to $C_p(X)$. In [4, p. 749], M. Lynch proved the following.

Theorem 2.3. *Let X be a continuum, μ a Whitney map for $C(X)$ and $p \in X$. Then $\mu_p^{-1}(t)$ is an AR for each $t \in [0, \mu(X)]$.*