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A note on multiplier convergent series

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ABSTRACT

Given a topological ring R and $\mathscr{F} \subset R^{\mathbb{N}}$ a (formal) series $\sum_{n\in\mathbb{N}} x_n$ in a topological R-module E is $\mathscr{F}$ *multiplier convergent in* E (respectively $\mathscr{F}$ *multiplier Cauchy in* E) provided that the sequence $\{\sum_{i=0}^{n} r(i)x_i : n \in \mathbb{N}\}$ of partial sums converges (respectively, is a Cauchy sequence) for every sequence function $r \in \mathscr{F}$. In this paper we investigate for which $\mathscr{G} \subset R^{\mathbb{N}}$ every $\mathscr{F}$ multiplier convergent (Cauchy) series is also $\mathscr{G}$ multiplier convergent (Cauchy). We obtain some general theorems about the Cauchy version of this problem. In particular, we prove that every $\mathbb{Z}^{\mathbb{N}}$ multiplier Cauchy series is already $\mathbb{R}^{\mathbb{N}}$ multiplier Cauchy in every topological vector space. On the other hand, we construct examples that in particular show that a $\mathbb{Z}^{\mathbb{N}}$ multiplier convergent series need not to be even $\mathbb{Q}^{\mathbb{N}}$ multiplier convergent and that there are topological vector spaces containing non-trivial $\mathbb{Q}^{\mathbb{N}}$ multiplier convergent series that do not contain non-trivial $\mathbb{R}^{\mathbb{N}}$ convergent series. As a consequence of this example, there are topological vector spaces containing the topological group $\mathbb{Q}^{\mathbb{N}}$ (and thus $\mathbb{Z}^{\mathbb{N}}$ and $\mathbb{Z}^{(\mathbb{N})}$ as well) that do not contain the topological vector space $\mathbb{R}^{\mathbb{N}}$. On the contrary, it was proved in [3], that a sequentially complete topological vector space that contains the topological group $\mathbb{Z}^{(\mathbb{N})}$ must already contain the topological vector space $\mathbb{R}^{\mathbb{N}}$. Hence our example demonstrates, that in the latter result, the condition of sequential completeness can not be weakened by assuming that the space in question contains the topological group $\mathbb{Z}^{\mathbb{N}}$ (which is the sequential completion of $\mathbb{Z}^{(\mathbb{N})}$).

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ stay for the sets of natural numbers, integers, rational numbers and real numbers respectively equipped with their usual algebraic and topological structures and $\mathfrak{c}$ denotes the cardinality of the continuum.

Every (topological) abelian group can be considered in a natural way as a (topological) $\mathbb{Z}$-module while every real (topological) vector space is an $\mathbb{R}$-module. In order to unify our approach for abelian (topological) groups and (topological) vector spaces, we will use the following notation based on R-modules with the (topological) ring R being mostly either the (topological) ring $\mathbb{Z}$ or the (topological) field $\mathbb{R}$. Let R be a ring with identity and M a left R-module. By 0_M we denote the zero element of M with respect to the

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group operation on M, and the subscript is often omitted when there is no danger of confusion. Given a subset A of M, the symbol $\langle A \rangle_R$ stays for the R-submodule of M generated by A. Further, by $R^{\mathbb{N}}$ we denote the direct product of $|\mathbb{N}|$-many copies of R, while $R^{(\mathbb{N})}$ stays for the direct sum of $|\mathbb{N}|$-many copies of R. That is, $R^{\mathbb{N}}$ consists of all functions $r : \mathbb{N} \to R$, while $R^{(\mathbb{N})}$ consists of all functions $r : \mathbb{N} \to R$ having finite support (that is, $r(i) \neq 0_R$ only for finitely many $i \in \mathbb{N}$). The sets $R^{\mathbb{N}}$ and $R^{(\mathbb{N})}$ become R-modules under the coordinate-wise addition and scalar multiplication. When R is a topological ring, we will consider $R^{\mathbb{N}}$ as well as its R-submodule $R^{(\mathbb{N})}$ with the Tychonoff product topology. A sequence in an R-module M is called *trivial* if all but finitely many elements of the sequence are equal to 0_M, otherwise it is called *non-trivial*.

1. Introduction

Let us start with an extension of the notion of a multiplier convergent series in a topological vector space [11, see Definition 2.1].

Definition 1.1. Let R be a topological ring and $\mathscr{F}$ a subset of the R-module $R^{\mathbb{N}}$. We say that a (formal) series $\sum_{n\in\mathbb{N}} x_n$ in a topological R-module E is *$\mathscr{F}$ multiplier convergent in E* (respectively *$\mathscr{F}$ multiplier Cauchy in E*) provided that the sequence $\{\sum_{i=0}^{n} r(i)x_i : n \in \mathbb{N}\}$ of partial sums converges (respectively, is a Cauchy sequence) for every sequence function $r \in \mathscr{F}$.

For brevity, we shall also say that the sequence $\{x_n : n \in \mathbb{N}\}$ is *$\mathscr{F}$-summable* (respectively, *$\mathscr{F}$-Cauchy summable*) if the series $\sum_{n=0}^{\infty} x_n$ is $\mathscr{F}$ multiplier convergent in E (respectively, $\mathscr{F}$ multiplier Cauchy in E).

The elements of $\mathscr{F}$ are called multipliers.

$\mathscr{F}$-(Cauchy) summable sequences, in the case when $R = \mathbb{Z}$, were studied extensively in [2] for a broad class of sets of multipliers. The toughest summability condition on a sequence is clearly imposed by taking $\mathscr{F}$ to be the whole $\mathbb{Z}^{\mathbb{N}}$. Topological groups without non-trivial $\mathbb{Z}^{\mathbb{N}}$-summable sequences were introduced in [5]. They were used under the name TAP groups in [9] to characterize pseudocompactness of a topological space X in terms of properties of the topological group $C_p(X, G)$ of all continuous G-valued functions on X endowed with the topology of pointwise convergence for a given topological group G ([9, see Theorem 6.5]). TAP groups were also studied in [2,4] where it was proved that a metric group contains no non-trivial $\mathbb{Z}^{\mathbb{N}}$-Cauchy summable sequences if and only if it is NSS (recall that an NSS group is a topological group which has a neighbourhood of the identity that contains no non-trivial subgroup). For more properties of non-abelian TAP groups see also [10]. It follows from Lemma 3.1 that the range of values of a $\mathbb{Z}^{\mathbb{N}}$-Cauchy summable sequence is *absolutely Cauchy summable* in the sense of [3, Definition 3.1]. This combined with [3, Theorem 11.1] gives us the following fact that demonstrates another important aspect of $\mathbb{Z}^{\mathbb{N}}$-Cauchy summable sequences.

Fact 1.2. In a sequentially complete topological vector space E the following conditions are equivalent:

(i) E contains a non-trivial $\mathbb{Z}^{\mathbb{N}}$-Cauchy summable sequence;
(ii) E contains the topological group $\mathbb{Z}^{(\mathbb{N})}$;
(iii) E contains the topological vector space $\mathbb{R}^{\mathbb{N}}$.

From [3, Corollary 7.4] we obtain the following property of $\mathbb{Z}^{\mathbb{N}}$-summable sequences.

Fact 1.3. A topological vector space contains the topological group $\mathbb{Z}^{\mathbb{N}}$ if and only if it contains a non-trivial $\mathbb{Z}^{\mathbb{N}}$-summable sequence.

In the case when $R = \mathbb{R}$ we refer to the monography [11] for more about the rich theory of multiplier convergent series. Similarly as in the previous case, the strongest summability condition one can impose on a