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The Ascoli property for function spaces

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ABSTRACT

The paper deals with Ascoli spaces $C_p(X)$ and $C_k(X)$ over Tychonoff spaces X. The class of Ascoli spaces X, i.e. spaces X for which any compact subset \mathcal{K} of $C_k(X)$ is evenly continuous, essentially includes the class of $k_{\mathbb{R}}$ -spaces. First we prove that if $C_p(X)$ is Ascoli, then it is κ -Fréchet–Urysohn. If X is cosmic, then $C_p(X)$ is Ascoli iff it is κ -Fréchet–Urysohn. This leads to the following extension of a result of Morishita: If for a Čech-complete space X the space $C_p(X)$ is Ascoli, then X is scattered. If X is a complete metrizable space, then $C_p(X)$ is Ascoli iff X is scattered and stratifiable, then $C_p(X)$ is Ascoli iff X is scattered. (b) If X is a Cech-complete Lindelöf space, then $C_p(X)$ is Ascoli iff X is scattered iff $C_p(X)$ is Fréchet–Urysohn. Moreover, we prove that for a paracompact space X of point-countable type the following conditions are equivalent: (i) X is locally compact. (ii) $C_k(X)$ is a $k_{\mathbb{R}}$ -space. (iii) $C_k(X)$ is an Ascoli space. The Ascoli spaces $C_k(X, \mathbb{I})$ are also studied.

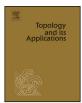
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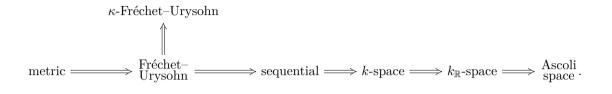
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1. Introduction

Various topological properties generalizing metrizability have been intensively studied both by topologists and analysts for a long time, and the following diagram gathers some of the most important concepts:



Note that none of these implications is reversible. The study of the above concepts for the function spaces with various topologies has a rich history and is also nowadays an active area of research, see [2,15,18,30] and references therein.

For Tychonoff topological spaces X and Y, we denote by $C_k(X, Y)$ and $C_p(X, Y)$ the space C(X, Y) of all continuous functions from X into Y endowed with the compact-open topology or the pointwise topology, respectively. If $Y = \mathbb{R}$, we shall write $C_k(X)$ and $C_p(X)$, respectively.

It is well-known that $C_p(X)$ is metrizable if and only if X is countable. Pytkeev, Gerlitz and Nagy (see §3 of [2]) characterized spaces X for which $C_p(X)$ is Fréchet–Urysohn, sequential or a k-space (these properties coincide for the spaces $C_p(X)$). Sakai in [26] described all spaces X for which $C_p(X)$ is κ -Fréchet–Urysohn, see Theorem 2.3 below. However, very little is known about spaces X for which $C_p(X)$ is an Ascoli space or a $k_{\mathbb{R}}$ -space.

Following [3], a space X is called an Ascoli space if each compact subset \mathcal{K} of $C_k(X)$ is evenly continuous, that is, the map $X \times \mathcal{K} \ni (x, f) \mapsto f(x) \in \mathbb{R}$ is continuous. Equivalently, X is Ascoli if the natural evaluation map $X \hookrightarrow C_k(C_k(X))$ is an embedding, see [3]. Recall that a space X is called a $k_{\mathbb{R}}$ -space if a real-valued function f on X is continuous if and only if its restriction $f|_K$ to any compact subset K of X is continuous. It is known that every $k_{\mathbb{R}}$ -space is Ascoli, but the converse is in general not true, see [3].

The class of Ascoli spaces was introduced in [3]. The question for which spaces X the space $C_p(X)$ is Ascoli or a $k_{\mathbb{R}}$ -space is posed in [10]. It turned out that for spaces of the form $C_p(X)$, the Ascoli property is formally stronger than the κ -Fréchet–Urysohn one. This follows from the following

Theorem 1.1.

- (i) If $C_p(X)$ is Ascoli, then it is κ -Fréchet-Urysohn.
- (ii) If $C_p(X)$ is κ -Fréchet-Urysohn and every compact $K \subset C_k(C_p(X))$ is first-countable, then $C_p(X)$ is Ascoli.

Recall that a regular space X is *cosmic* if it is a continuous image of a separable metrizable space, see [19]. Michael proved in [19] that every compact subset of a cosmic space is metrizable, and if X is a cosmic space then $C_p(X)$ and hence $C_p(C_p(X))$ are cosmic. So all compact subsets of $C_p(C_p(X))$ and hence $C_k(C_p(X))$ are metrizable. This remark and Theorem 1.1 imply

Corollary 1.2. If X is a cosmic space, then $C_p(X)$ is Ascoli if and only if it is κ -Fréchet-Urysohn.

The second main result of Section 2 is the following theorem, which extends an unpublished result of Morishita [14, Theorem 10.7] and [6, Corollary 4.2], see also Corollary 2.12 below.

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