Topology and its Applications 214 (2016) 35–50

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

The Ascoli property for function spaces

Saak Gabriyelyan [a,*], Jan Grebík [b,1], Jerzy Kąkol [c,b,2], Lyubomyr Zdomskyy [d,3]

[a] Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva, P.O. 653, Israel
[b] Institute of Mathematics, Czech Academy of Sciences, Czech Republic
[c] A. Mickiewicz University, 61-614 Poznań, Poland
[d] Kurt Gödel Research Center for Mathematical Logic, University of Vienna, Währinger Straße 25, A-1090 Wien, Austria

A R T I C L E I N F O

Article history:
Received 19 August 2016
Received in revised form 26 August 2016
Accepted 30 August 2016
Available online 3 September 2016

MSC:
54C35
54D50

Keywords:
$C_p(X)$
$C_k(X)$
Ascoli
κ-Fréchet–Urysohn
Scattered
Čech-complete
Stratifiable
Paracompact

A B S T R A C T

The paper deals with Ascoli spaces $C_p(X)$ and $C_k(X)$ over Tychonoff spaces X. The class of Ascoli spaces X, i.e. spaces X for which any compact subset $\mathcal{K}$ of $C_k(X)$ is evenly continuous, essentially includes the class of $k_{\mathbb{R}}$-spaces. First we prove that if $C_p(X)$ is Ascoli, then it is κ-Fréchet–Urysohn. If X is cosmic, then $C_p(X)$ is Ascoli iff it is κ-Fréchet–Urysohn. This leads to the following extension of a result of Morishita: If for a Čech-complete space X the space $C_p(X)$ is Ascoli, then X is scattered. If X is scattered and stratifiable, then $C_p(X)$ is an Ascoli space. Consequently: (a) If X is a complete metrizable space, then $C_p(X)$ is Ascoli iff X is scattered. (b) If X is a Čech-complete Lindelöf space, then $C_p(X)$ is Ascoli iff X is scattered iff $C_p(X)$ is Fréchet–Urysohn. Moreover, we prove that for a paracompact space X of point-countable type the following conditions are equivalent: (i) X is locally compact. (ii) $C_k(X)$ is a $k_{\mathbb{R}}$-space. (iii) $C_k(X)$ is an Ascoli space. The Ascoli spaces $C_k(X, \mathbb{I})$ are also studied.

* Corresponding author.
E-mail addresses: saak@math.bgu.ac.il (S. Gabriyelyan), Greboshrabos@seznam.cz (J. Grebík), kakol@amu.edu.pl (J. Kąkol), lzdomsky@gmail.com (L. Zdomskyy).

1 The second author was supported by the GACR project 15-34700L and RVO: 67985840.
2 The third author was supported by Generalitat Valenciana, Conselleria d'Educació, Cultura i Esport, Spain, Grant PROMETEO/2013/058 and by the GAČR project 16-34860L and RVO: 67985840, and gratefully acknowledges also the financial support he received from the Kurt Goedel Research Center in Wien for his research visit in days 15.04–24.04 2016.
3 The fourth author would like to thank the Austrian Science Fund FWF (Grant I 1209-N25) for generous support for this research. The collaboration of the second and the fourth authors was partially supported by the Czech Ministry of Education grant 7AMB15AT035 and RVO: 67985840.

http://dx.doi.org/10.1016/j.topol.2016.08.026
0166-8641/

1. Introduction

Various topological properties generalizing metrizability have been intensively studied both by topologists and analysts for a long time, and the following diagram gathers some of the most important concepts:

$$\begin{tikzcd}
 & \kappa\text{-Fréchet–Urysohn} & & & & \\
\text{metric} \arrow[r, Rightarrow] & \begin{array}{c}\text{Fréchet–}\\ \text{Urysohn}\end{array} \arrow[u, Rightarrow] \arrow[r, Rightarrow] & \text{sequential} \arrow[r, Rightarrow] & k\text{-space} \arrow[r, Rightarrow] & k_{\mathbb{R}}\text{-space} \arrow[r, Rightarrow] & \begin{array}{c}\text{Ascoli}\\ \text{space}\end{array}.
\end{tikzcd}$$

Note that none of these implications is reversible. The study of the above concepts for the function spaces with various topologies has a rich history and is also nowadays an active area of research, see [2,15,18,30] and references therein.

For Tychonoff topological spaces X and Y, we denote by $C_k(X,Y)$ and $C_p(X,Y)$ the space $C(X,Y)$ of all continuous functions from X into Y endowed with the compact-open topology or the pointwise topology, respectively. If $Y = \mathbb{R}$, we shall write $C_k(X)$ and $C_p(X)$, respectively.

It is well-known that $C_p(X)$ is metrizable if and only if X is countable. Pytkeev, Gerlitz and Nagy (see §3 of [2]) characterized spaces X for which $C_p(X)$ is Fréchet–Urysohn, sequential or a k-space (these properties coincide for the spaces $C_p(X)$). Sakai in [26] described all spaces X for which $C_p(X)$ is κ-Fréchet–Urysohn, see Theorem 2.3 below. However, very little is known about spaces X for which $C_p(X)$ is an Ascoli space or a $k_{\mathbb{R}}$-space.

Following [3], a space X is called an *Ascoli space* if each compact subset $\mathcal{K}$ of $C_k(X)$ is evenly continuous, that is, the map $X \times \mathcal{K} \ni (x,f) \mapsto f(x) \in \mathbb{R}$ is continuous. Equivalently, X is Ascoli if the natural evaluation map $X \hookrightarrow C_k(C_k(X))$ is an embedding, see [3]. Recall that a space X is called a $k_{\mathbb{R}}$*-space* if a real-valued function f on X is continuous if and only if its restriction $f|_K$ to any compact subset K of X is continuous. It is known that every $k_{\mathbb{R}}$-space is Ascoli, but the converse is in general not true, see [3].

The class of Ascoli spaces was introduced in [3]. The question for which spaces X the space $C_p(X)$ is Ascoli or a $k_{\mathbb{R}}$-space is posed in [10]. It turned out that for spaces of the form $C_p(X)$, the Ascoli property is formally stronger than the κ-Fréchet–Urysohn one. This follows from the following

Theorem 1.1.

(i) *If $C_p(X)$ is Ascoli, then it is κ-Fréchet–Urysohn.*
(ii) *If $C_p(X)$ is κ-Fréchet–Urysohn and every compact $K \subset C_k(C_p(X))$ is first-countable, then $C_p(X)$ is Ascoli.*

Recall that a regular space X is *cosmic* if it is a continuous image of a separable metrizable space, see [19]. Michael proved in [19] that every compact subset of a cosmic space is metrizable, and if X is a cosmic space then $C_p(X)$ and hence $C_p(C_p(X))$ are cosmic. So all compact subsets of $C_p(C_p(X))$ and hence $C_k(C_p(X))$ are metrizable. This remark and Theorem 1.1 imply

Corollary 1.2. *If X is a cosmic space, then $C_p(X)$ is Ascoli if and only if it is κ-Fréchet–Urysohn.*

The second main result of Section 2 is the following theorem, which extends an unpublished result of Morishita [14, Theorem 10.7] and [6, Corollary 4.2], see also Corollary 2.12 below.