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Large scale absolute extensors

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ABSTRACT

This paper is devoted to dualization of dimension-theoretical results from the small scale to the large scale. So far there are two approaches for such dualization: one consisting of creating analogs of small scale concepts using covers and the other amounting to the covering dimension of the Higson corona $\nu(X)$ of X. The first approach was used by M. Gromov when defining the asymptotic dimension $\mathrm{asdim}(X)$ of metric spaces X. The second approach was implicitly contained in the paper [6] by Dranishnikov on asymptotic topology. It is not known if the two approaches yield the same concept. However, Dranishnikov–Keesling–Uspenskiy proved $\dim(\nu(X)) \le \mathrm{asdim}(X)$ and Dranishnikov established that $\dim(\nu(X)) = \mathrm{asdim}(X)$ provided $\mathrm{asdim}(X) < \infty$. We characterize asymptotic dimension (for spaces of finite asymptotic dimension) in terms of extensions of slowly oscillating functions to spheres. Our approach is specifically designed to relate asymptotic dimension to the covering dimension of the Higson corona $\nu(X)$ in case of proper metric spaces X. As an application, we recover the results of Dranishnikov–Keesling–Uspenskiy and Dranishnikov.

1. Introduction

Asymptotic dimension of metric spaces was introduced by M. Gromov [10] as a means of exploring large scale properties of the space and has been studied extensively during the last two decades. Gromov's definition (see 3.1) dualizes covering dimension as follows: instead of refining open covers by open covers of multiplicity at most $n+1$, it asks for coarsening of uniformly bounded covers $\mathcal{U}$ by uniformly bounded covers $\mathcal{V}$ with the property that every element U of $\mathcal{U}$ intersects at most $n+1$ elements of $\mathcal{V}$.

In case of proper metric spaces X (that means bounded subsets of X have compact closure) there is another way to introduce a coarse invariant related to dimension. Namely, it is the covering dimension $\dim(\nu(X))$ of the Higson corona $\nu(X)$ of X (see below). As shown in [18], two proper metric spaces that are coarsely equivalent have homeomorphic Higson coronas, hence their covering dimensions are the same.

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Thus, indeed, $\dim(\nu(X))$ is a coarse invariant and it is an open question if it is equal to the asymptotic dimension of X. This paper is devoted to the internal characterization of $\dim(\nu(X))$, a characterization that makes sense to all metric spaces, not just proper metric spaces.

We know (see [7]) that for a proper metric space X, the covering dimension of the Higson corona does not exceed the asymptotic dimension of X. Also, see [6], $\dim(\nu(X)) = \mathrm{asdim}(X)$ if X is a proper metric spaces of finite asymptotic dimension $\mathrm{asdim}(X)$. Our approach gives alternative proofs of those results.

Recall that the Higson corona $\nu(X)$ is the complement of X in its Higson compactification $h(X)$. In turn, the Higson compactification $h(X)$ is characterized by the fact that all continuous slowly oscillating functions from X to the unit interval $[0,1]$ extend to continuous functions on $h(X)$. Thus, it is an analog of the Čech–Stone compactification where the family of all continuous functions is replaced by all continuous and slowly oscillating functions. The simplest definition of $f : X \to [0,1]$ being **slowly oscillating** is that $|f(x_n) - f(y_n)| \to 0$ whenever $\sup(d(x_n, y_n)) < \infty$ and $x_n \to \infty$ (that means each bounded subset of X contains only finitely many elements of the sequence $\{x_n\}_{n\geq 1}$). In particular, every function of compact support is slowly oscillating, hence the Higson compactification of X does exist and contains X topologically.

Since covering dimension $\dim(X)$ of a compact space X being at most n can be characterized by saying that the n-sphere S^n is an absolute extensor of X (that means any continuous map $f : A \to S^n$, A being a closed subset of X, can be extended over X), one should look for analogous concept involving slowly oscillating functions (not necessarily being continuous). Therefore we introduce the concept of a **large scale absolute extensor** of a metric space. K is a **large scale absolute extensor of** X ($K \in \text{ls-AE}(X)$) if for any subset A of X and any slowly oscillating function $f\colon A \to K$ there is a slowly oscillating extension $g\colon X \to K$. We characterize large scale absolute extensors of a space in terms of extensions of (ϵ, R)-continuous functions. It turns out that being large scale absolute extensor of a space is a coarse invariant of the space. In the later part of the paper we find necessary and sufficient conditions for a sphere S^m to be a large scale extensor of X. This is done by comparing existence of Lipschitz extensions in a finite range of Lipschitz constants to existence of Lebesgue refinements in a finite range of Lebesgue constants. We characterize asymptotic dimension of the space in terms of spheres being large scale absolute extensors of the space.

Another natural idea is to study large scale extensors Y of a metric space X defined as follows:

For any $\epsilon > 0$ there is $\delta > 0$ such that any (δ, δ)-Lipschitz map $f\colon A \subset X \to Y$ extends to an (ϵ, ϵ)-Lipschitz map $g\colon X \to Y$. It turns out, for bounded metric spaces Y, this approach is equivalent to the one involving slowly oscillating functions.

There have been two earlier approaches where connections of asymptotic dimension with extensions of other categories of maps have been studied. One is A. Dranishnikov's ([6], using extensions of proper asymptotically Lipschitz functions to euclidean spaces), and the other is due to Repovš–Zarichnyi ([17], using maps to open cones). In a separate paper [9] we examine how those existing concepts relate to our concept of large scale absolute extensors.

1.1. *C^*-algebra approach to asymptotic dimension*

There is another way to create large scale analogs of covering dimension by using the concept of the **nuclear dimension** of C*-algebras (see [21]). Given any functor F from the large scale category to the category of C*-algebras, one can consider the nuclear dimension of $F(X)$ for any coarse space X and it gives rise to a large scale invariant. So far we know of two useful functors F; one is $\frac{B_h(X)}{B_0(X)}$ (see p. 31 of [18]), where $B_h(X)$ is the algebra of all bounded functions $X \to \mathbb{C}$ that are slowly oscillating and $B_0(X)$ is the subalgebra of $B_h(X)$ consisting of functions that tend to 0 at infinity. The second useful C*-algebra of a coarse space X is its uniform Roe algebra (see Chapter 4 of [18]).

The algebra $\frac{B_h(X)}{B_0(X)}$ being unital and commutative, has a compact spectrum whose covering dimension coincides with the nuclear dimension of $\frac{B_h(X)}{B_0(X)}$ (see [21]). In case of proper metric spaces X, the spectrum of $\frac{B_h(X)}{B_0(X)}$ is exactly the Higson corona $\nu(X)$ of X. Thus, one can view the spectrum of $\frac{B_h(X)}{B_0(X)}$ as an abstract