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Is a monotone union of contractible open sets contractible?

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This paper presents some partial answers to the following question.

Question. If a normal space X is the union of an increasing sequence of open sets $U_1 \subset U_2 \subset U_3 \subset \ldots$ such that each U_n contracts to a point in X, must X be contractible?

The main results of the paper are:

Theorem 1. *If a normal space X is the union of a sequence of open subsets $\{U_n\}$ such that $cl(U_n) \subset U_{n+1}$ and U_n contracts to a point in U_{n+1} for each $n \geq 1$, then X is contractible.*

Corollary 2. *If a locally compact σ-compact normal space X is the union of an increasing sequence of open sets $U_1 \subset U_2 \subset U_3 \subset \ldots$ such that each U_n contracts to a point in X, then X is contractible.*

1. Introduction

In 1935, J. H. C. Whitehead, to illustrate a flaw in his own proposed proof of the Poincaré Conjecture, constructed a contractible open[1] 3-manifold without boundary that is not homeomorphic to $\mathbb{R}^3$ [7]. Subsequently it was shown that in each dimension $n \geq 3$, there exist uncountably many non-homeomorphic contractible open n-manifolds. (See [5], [1] and [3].) These spaces illustrate the richness of the topology of manifolds in dimensions greater than 2.

Proofs that a construction yields a contractible open n-manifold that is not homeomorphic to $\mathbb{R}^n$ characteristically have two steps. First they establish that the constructed space is contractible. Second they

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[1] A manifold is called *open* if it is non-compact and has empty boundary.

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show that it is not homeomorphic to $\mathbb{R}^n$. While the second step is usually the more interesting and delicate of the two, in this article we focus on methods used to take the first step.

Typical constructions of contractible open manifolds produce a space X that is the union of an increasing sequence of open subsets $U_1 \subset U_2 \subset U_3 \subset \ldots$ such that each U_n contracts to a point in X. With this information one can justify the contractibility of X in various ways. For instance, if X is a CW complex, then one can observe that all the homotopy groups of X vanish and use a theorem of J. H. C. Whitehead (Corollary 24 on page 405 of [6]) to conclude that X is contractible. If a more elementary justification is sought which avoids assuming that the space X is a CW complex or appealing to the theorem of Whitehead, then the following theorem provides an approach.

Theorem 3. *If a normal space X is the union of a sequence of open subsets $\{U_n\}$ and there is a point $p_0 \in U_1$ such that for each $n \geq 1, cl(U_n) \subset U_{n+1}$ and U_n contracts to p_0 in U_{n+1} fixing p_0, then X is contractible.*

The proof of Theorem 3 is elementary and well known. Observe that Theorem 3 follows immediately from Theorem 1. (Also the first half of the proof of Theorem 1 given below is essentially a proof of Theorem 3. A parenthetical comment in the proof of Theorem 1 marks the point at which the proof of Theorem 3 is complete.) Applying Theorem 3 directly to a space X requires some care in the construction of X to insure that each U_n contracts to an initially specified point p_0 in U_{n+1} fixing the point p_0. The motivation behind this paper is to show that we can weaken the hypotheses of Theorem 3 to those of Theorem 1 and thereby remove the requirement that the homotopy contracting U_n to a point in U_{n+1} fixes any particular point. As a consequence, in the construction of a contractible open manifold, the argument that the constructed object is contractible becomes easier while still relying on principles that are valid in a very broad setting (the realm of normal spaces).

We remark that the hypothesis that the homotopy contracting U_n to a point in U_{n+1} fix the point can't be dropped with impunity because there exist contractible metric spaces that can't be contracted to a point fixing that point. The *line of Cantor fans* is a simple non-compact example of such a space. This space is the countable union $\cup_{n \in \mathbb{Z}} K_n$ in which K_n is the cone in the plane with vertex $(n, 0)$ and base $\{n+1\} \times C$ where C is the standard middle-thirds Cantor set in $[0, 1]$. A more complex compact example is the *Cantor sting ray* described in [2]. (A comparable complete description of the Cantor sting ray can be found in Exercise 7 on pages 18–19 in [4].)

Although the requirement that the contracting homotopies fix a point can't be omitted without consequence, it is known that it can be omitted if one is willing to impose additional conditions on X as in the following result.

Theorem 4. *If a normal space X is the union of a sequence of open subsets $\{U_n\}$ such that for each $n \geq 1$, $cl(U_n) \subset U_{n+1}$ and U_n contracts to point in U_{n+1}, then X is contractible provided that it satisfies the following additional condition.*

$(*)$ *There is an open subset V of X that contracts to a point $p_0 \in V$ in X fixing p_0.*

Theorem 4 follows from Theorem 3 and the following lemma.

Lemma 5. *If $W \subset U_1 \subset U_2$ are open subsets of a completely regular space X and if W contracts to a point $p_0 \in W$ in U_1 fixing p_0 and U_1 contracts to a point in U_2, then U_1 contracts to p_0 in U_2 fixing p_0.*

Although the proof of Lemma 5 is known and is similar to the proofs of Theorem 1.4.11 on pages 31 and 32 and Exercise 1.D.4 on page 57 of [6], we follow the referee's recommendation that we include a proof.