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On metric orbit spaces and metric dimension

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ABSTRACT

For a metric space (X, d), a subset A resolves (X, d) if each point $x \in X$ is uniquely determined by the distances $d(x, a)$ for $a \in A$. Also the metric dimension of (X, d) is the smallest cardinality $md(X)$ such that there is a set A of the cardinality $md(X)$ that resolves X.

In this note we are going to determine the metric dimension of metric orbit spaces in some special cases and find an upper bound for a general case. This category contains a vast domain of topological spaces and topological manifolds.

1. Introduction

In 1953, Blumental [2] for the first time introduced the concept of the metric dimension of a metric space. The concept received more attention by way of its application in the set of the vertices of a graph (e.g. [7,14]). Since then it has found further applications in many other disciplines (e.g. [3–5,10,12]). Bau and Beardon [1], returning to the original idea of the metric dimension of a metric space, computed among other things, the metric dimension for n-dimensional Euclidean space, spherical space, hyperbolic space, and Riemann surfaces. Recently, we [8] presented some generalizations of [1] and computed the metric dimensions of n-dimensional geometric spaces. (See [9] where the metric dimension for the metric manifolds has been computed.) In [6] using metric dimension, an interesting characterization of the points in a simplex for a normed space has been given.

Let us recall from [1] that for a metric space (X, d) by a resolving set we mean a non-empty subset A of X with if $d(x, a) = d(y, a)$ for all $a \in A$ then $x = y$. The *metric dimension* $md(X)$ of (X, d) is the smallest

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cardinality κ such that there is a resolving subset of X with the cardinality κ. A subset of (X, d) with cardinality $md(X)$ that resolves X is called a *metric basis* for X. As X resolves X every metric space X has a metric dimension which is at most the cardinality $|X|$ of X.

In this note our aim is to determine the metric dimension of a class of metric spaces called metric orbit spaces. A vast domain of topological spaces and topological manifolds fall in this class. This will give us a tool to compute the metric dimension for more general cases of metric spaces.

2. Preliminaries

In this section we present some preliminary definitions. Our definitions and notation concerning topological and metric spaces and manifolds are standard; see, for example, [11].

As usual we define Euclidean space, hyperbolic space, and spherical space, respectively, by

$$\mathbb{E}^n = \{x = (x_1, ..., x_n) \mid x_i \in \mathbb{R}\} \text{ with the metric } d(x, y) = \|x - y\|$$

$$\mathbb{H}^n = \{x \in \mathbb{R}^n \mid x_n > 0\} \text{ with the path metric derived from } |dx|/x_n$$

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\} \text{ with the path metric induced by the Euclidean metric on } \mathbb{R}^{n+1}.$$

We need the following fact from [1].

Lemma 2.1. *Suppose $X = \mathbb{E}^n, \mathbb{H}^n, \mathbb{S}^n$, or any open subset of $\mathbb{E}^n$, then $md(X) = n + 1$.*

Let us remark that in a metric space X, the relation $A \subseteq B \subseteq X$ does not, in general, imply neither $md(A) \leq md(B)$ nor $md(B) \leq md(A)$, see [8].

By an *n-dimensional geometric space* we mean a metric space (M, d) that is an n dimensional connected homogeneous Riemannian manifold. For example the spaces $\mathbb{E}^n$, $\mathbb{H}^n$, $\mathbb{S}^n$, $\mathbb{T}^n$ (n-Torus), $\mathbb{RP}^n$ (the real n-projective space), and $\mathbb{CP}^n$ (the complex n-projective space) real Grassmannian $O(n)/(O(r) \times O(n - r))$ and complex Grassmannian $U(n)/(U(r) \times U(n - r))$ manifolds are elementary examples of geometric spaces. Let us remark that from [8] we know that for an n-dimensional geometric space X, $md(X) = n + 1$. For the main and equivalent definitions of a geometric space, see [13].

Definition 2.2.

(i) Let G be a subgroup of $S(X)$, the similarity group of an n-dimensional geometric space X and let M be an n-manifold. An (X, G)-atlas for M is defined as a family of charts

$$\Phi = \{\phi_i : U_i \to X \mid i \in I\}$$

covering M such that the coordinate changes

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$$

agree in a neighborhood of each point with an element of G. There is a unique maximal (X, G)-atlas for M containing Φ. An (X, G)-structure for M is a maximal (X, G)-atlas for M and an (X, G)-*manifold* is an n-manifold M together with an (X, G)-structure for M.

(ii) A *metric* (X, G)-*manifold* is a connected (X, G)-manifold M such that G is a subgroup of $I(X)$, the group of isometries of X.