Topology and its Applications 214 (2016) 109–126

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On the topology of real bundle pairs over nodal symmetric surfaces

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ARTICLE INFO

Article history:
Received 14 September 2016
Accepted 6 October 2016
Available online 13 October 2016

Keywords:
Classification
Real bundle pairs
Symmetric surfaces

ABSTRACT

We give an alternative argument for the classification of real bundle pairs over smooth symmetric surfaces and extend this classification to nodal symmetric surfaces. We also classify the homotopy classes of automorphisms of real bundle pairs over symmetric surfaces. The two statements together describe the isomorphisms between real bundle pairs over symmetric surfaces up to deformation.

1. Introduction

The study of symmetric surfaces goes back to at least [10]. They have since played important roles in different areas of mathematics, as indicated by [1] and its citations. Real bundle pairs, or Real vector bundles in the sense of [2], over smooth symmetric surfaces are classified in [3]. In this paper, we give an alternative proof of this core result of [3], obtain its analogue for nodal symmetric surfaces, and classify the automorphisms of real bundle pairs over symmetric surfaces. Special cases of the main results of this paper, Theorems 1.1 and 1.2 below, are one of the ingredients in the construction of positive-genus real Gromov–Witten invariants in [8] and in the study of their properties in [9].

An involution on a topological space X is a homeomorphism $\phi : X \longrightarrow X$ such that $\phi \circ \phi = \mathrm{id}_X$. A symmetric surface (Σ, σ) is a closed oriented (possibly nodal) surface Σ with an orientation-reversing involution σ. If Σ is smooth, the fixed locus Σ^σ of σ is a disjoint union of circles. In general, Σ^σ consists of isolated points (called E nodes in [11, Section 3.2]) and circles identified at pairs of points (called H nodes in [11, Section 3.2]).

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[1] Supported by ERC grant STEIN-259118.
[2] Partially supported by NSF grant DMS 1500875 and MPIM.

http://dx.doi.org/10.1016/j.topol.2016.10.002
0166-8641/

Let (X,ϕ) be a topological space with an involution. A conjugation on a complex vector bundle $V \longrightarrow X$ lifting ϕ is a vector bundle homomorphism $\varphi\colon V \longrightarrow V$ covering ϕ (or equivalently a vector bundle homomorphism $\varphi\colon V \longrightarrow \phi^*V$ covering id_X) such that the restriction of φ to each fiber is anti-complex linear and $\varphi\circ\varphi = \mathrm{id}_V$. A real bundle pair $(V,\varphi) \longrightarrow (X,\phi)$ consists of a complex vector bundle $V \longrightarrow X$ and a conjugation φ on V lifting ϕ. For example,

$$(X\times\mathbb{C}^n, \phi\times\mathfrak{c}) \longrightarrow (X,\phi),$$

where $\mathfrak{c}\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is the standard conjugation on $\mathbb{C}^n$, is a real bundle pair; we call it the trivial rank n real bundle pair over (X,ϕ). For any real bundle pair (V,φ) over (X,ϕ), the fixed locus

$$V^\varphi \equiv \big\{v\in V\colon\ \varphi(v)=v\big\}$$

of φ is a real vector bundle over the fixed locus X^ϕ of ϕ with $\mathrm{rk}_{\mathbb{R}}V^\varphi = \mathrm{rk}_{\mathbb{C}}V$.

If (V_1,φ_1) and (V_2,φ_2) are real vector bundle pairs over (X,ϕ), an isomorphism

$$\Phi\colon (V_1,\varphi_1) \longrightarrow (V_2,\varphi_2) \tag{1.1}$$

of real bundle pairs over (X,ϕ) is a $\mathbb{C}$-linear isomorphism $\Phi\colon V_1 \longrightarrow V_2$ covering the identity id_X such that $\Phi\circ\varphi_1 = \varphi_2\circ\Phi$. We call two real bundle pairs (V_1,φ_1) and (V_2,φ_2) over (X,ϕ) isomorphic if there exists an isomorphism of real bundle pairs as in (1.1). Our first theorem classifies real bundle pairs over symmetric surfaces up to isomorphism.

Theorem 1.1. *Suppose (Σ,σ) is a (possibly nodal) symmetric surface. Two real bundle pairs (V_1,φ_1) and (V_2,φ_2) over (Σ,σ) are isomorphic if and only if*

$$\mathrm{rk}_{\mathbb{C}}V_1 = \mathrm{rk}_{\mathbb{C}}V_2, \qquad w_1\big(V_1^{\varphi_1}\big) = w_1\big(V_2^{\varphi_2}\big) \in H^1(\Sigma^\sigma;\mathbb{Z}_2),$$

and $\deg(V_1|_{\Sigma'}) = \deg(V_2|_{\Sigma'})$ *for each irreducible component* $\Sigma'\subset\Sigma$.

Let X be a topological space. We denote by $\mathcal{C}(X;\mathbb{R}^*)$ and $\mathcal{C}(X;\mathbb{C}^*)$ the topological groups of $\mathbb{R}^*$-valued and $\mathbb{C}^*$-valued, respectively, continuous functions on X. For a real vector bundle V over X, let $\mathrm{GL}(V)$ be the topological group of vector bundle isomorphisms of V with itself covering id_X and $\mathrm{SL}(V)\subset\mathrm{GL}(V)$ be the subgroup of isomorphisms ψ so that the induced isomorphism

$$\Lambda^{\mathrm{top}}_{\mathbb{R}}\psi\colon \Lambda^{\mathrm{top}}_{\mathbb{R}}V \longrightarrow \Lambda^{\mathrm{top}}_{\mathbb{R}}V$$

is the identity. If V is a line bundle, then $\mathrm{GL}(V)$ is naturally identified with $\mathcal{C}(X;\mathbb{R}^*)$ and $\mathrm{SL}(V)\subset\mathrm{GL}(V)$ is the one-point set consisting of the constant function 1. For an arbitrary real vector bundle V over X and $\psi\in\mathrm{GL}(V)$, we denote by $\det_{\mathbb{R}}\psi$ the continuous function on X corresponding to the isomorphism $\Lambda^{\mathrm{top}}_{\mathbb{R}}\psi$ of $\Lambda^{\mathrm{top}}_{\mathbb{R}}V$.

Let (X,ϕ) be a topological space with an involution. Denote by

$$\mathcal{C}\big(X,\phi;\mathbb{C}^*\big) \subset \mathcal{C}\big(X;\mathbb{C}^*\big)$$

the subgroup of continuous maps f such that $f(\phi(z)) = \overline{f(z)}$ for all $z\in X$. The restriction of such a function to the fixed locus $X^\phi\subset\Sigma$ takes values in $\mathbb{R}^*$, i.e. gives rise to a homomorphism

$$\mathcal{C}\big(X,\phi;\mathbb{C}^*\big) \longrightarrow \mathcal{C}\big(X;\mathbb{R}^*\big), \qquad f \longrightarrow f\big|_{X^\phi}.$$