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Connectedness modulo an ideal ☆

M.R. Koushesh [a,b,*]

[a] *Department of Mathematical Sciences, Isfahan University of Technology, Isfahan, 84156-83111, Iran*
[b] *School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Iran*

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A B S T R A C T

For a topological space X and an ideal $\mathscr{H}$ of subsets of X we introduce the notion of connectedness modulo $\mathscr{H}$. This notion of connectedness naturally generalizes the notion of connectedness in its usual sense. In the case when X is completely regular, we introduce a subspace $\gamma_{\mathscr{H}}X$ of the Stone–Čech compactification βX of X, such that connectedness modulo $\mathscr{H}$ is equivalent to connectedness of $\beta X \setminus \gamma_{\mathscr{H}}X$. In particular, we prove that when $\mathscr{H}$ is the ideal generated by the collection of all open subspaces of X with pseudocompact closure, then X is connected modulo $\mathscr{H}$ if and only if $\mathrm{cl}_{\beta X}(\beta X \setminus \upsilon X)$ is connected, and when X is normal and $\mathscr{H}$ is the ideal generated by the collection of all closed realcompact subspaces of X, then X is connected modulo $\mathscr{H}$ if and only if $\mathrm{cl}_{\beta X}(\upsilon X \setminus X)$ is connected. Here υX is the Hewitt realcompactification of X.

1. Introduction

Throughout this article by *completely regular* we mean completely regular and Hausdorff (that is, Tychonoff).

An *ideal* $\mathscr{H}$ in a set X is a non-empty collection of subsets of X such that

- if a set A is contained in an element of $\mathscr{H}$ then A is in $\mathscr{H}$,
- if G and H are in $\mathscr{H}$ then so is their union $G \cup H$.

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* Correspondence to: Department of Mathematical Sciences, Isfahan University of Technology, Isfahan, 84156-83111, Iran.
E-mail address: koushesh@cc.iut.ac.ir.

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Intuitively, an ideal is a collection of subsets that are considered to be "small". Let $\mathscr{A}$ be a collection of subsets of a set X. The *ideal in* X *generated by* $\mathscr{A}$, denoted by $\langle\mathscr{A}\rangle$, is the intersection of all ideals in X which contain $\mathscr{A}$. It is easy to check that $\langle\mathscr{A}\rangle$ is the collection of all subsets of finite unions of elements from $\mathscr{A}$.

The purpose of this article is to present a natural generalization of the notion of connectedness in topological spaces. Recall that a space X is said to be *connected* if there is no continuous 2-valued mapping on X, that is, there is no continuous mapping $f : X \to [0, 1]$ such that

- $f^{-1}(0)$ and $f^{-1}(1)$ are neither empty,
- $X \setminus (f^{-1}(0) \cup f^{-1}(1))$ is empty.

Now, to generalize, we replace "emptyness" in the above definition by "smallness", that is, "being an element of an ideal in X". More precisely, for a space X and an ideal $\mathscr{H}$ in X, we say that X *is connected modulo* $\mathscr{H}$ if there is no continuous mapping $f : X \to [0, 1]$ such that

- $f^{-1}(0)$ and $f^{-1}(1)$ are neither in $\mathscr{H}$,
- $X \setminus (f^{-1}(0) \cup f^{-1}(1))$ is in $\mathscr{H}$.

From this simple definition much follows. Indeed, most standard facts about connectedness have counterparts in this context. Note that for the trivial ideal $\{\emptyset\}$, connectedness modulo an ideal coincides with connectedness in its usual sense.

This article is organized as follows. There are two main sections. In Section 2 we study connectedness modulo an ideal $\mathscr{H}$ with no particular restriction on $\mathscr{H}$; in Section 3 we deal with particular examples of $\mathscr{H}$. In Section 2 we generalize the standard known facts about connectedness to our new setting. This include theorems on preservation of connectedness under continuous mappings, formation of unions and taking closures. Further, for a completely regular space X and an ideal $\mathscr{H}$ of subsets of X, we introduce a subspace $\lambda_{\mathscr{H}}X$ of the Stone–Čech compactification βX of X, such that connectedness modulo the ideal $\mathscr{H}$ of X is equivalent to connectedness of $\beta X \setminus \lambda_{\mathscr{H}}X$. In Section 3 we consider particular examples of the ideal $\mathscr{H}$. In particular, we show that if X is completely regular and $\mathscr{H}$ is the ideal generated by the collection of all open subspaces of X whose closures are pseudocompact, then X is connected modulo $\mathscr{H}$ if and only if $\mathrm{cl}_{\beta X}(\beta X \setminus \upsilon X)$ is connected, and if X is normal and $\mathscr{H}$ is the ideal generated by the collection of all closed realcompact subspaces of X, then X is connected modulo $\mathscr{H}$ if and only if $\mathrm{cl}_{\beta X}(\upsilon X \setminus X)$ is connected. Here υX denotes the Hewitt realcompactification of X.

We now review some notation and certain known facts. For undefined terms and notation we refer the reader to the standard texts [4,5] and [18].

Let X be a space. A *zero-set in* X (*cozero-set in* X, respectively) is a set of the form $f^{-1}(0)$ ($X \setminus f^{-1}(0)$, respectively) where $f : X \to [0, 1]$ is a continuous mapping. For a continuous mapping $f : X \to [0, 1]$, the *zero-set of* f (*cozero-set of* f, respectively) is $f^{-1}(0)$ ($X \setminus f^{-1}(0)$, respectively). The set of all zero-sets of X (cozero-sets of X, respectively) is denoted by $\mathrm{Z}(X)$ ($\mathrm{Coz}(X)$, respectively).

The Stone–Čech compactification. Let X be a completely regular space. By a *compactification of* X we mean a compact Hausdorff space which contains X as a dense subspace. The *Stone–Čech compactification of* X, denoted by βX, is the compactification of X which is characterized among all compactifications of X by the fact that every continuous mapping $f : X \to K$, where K is a compact Hausdorff space (or $[0, 1]$), is extendable to a (unique) continuous mapping over βX; we denote this continuous extension of f by f_β. The Stone–Čech compactification of a completely regular space always exists.

The Hewitt realcompactification. A space is called *realcompact* if it is homeomorphic to a closed subspace of some product $\mathbb{R}^\alpha$. Let X be a completely regular space. A *realcompactification of* X is a realcompact