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Concordance of certain 3-braids and Gauss diagrams

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A B S T R A C T

Let $\beta := \sigma_1\sigma_2^{-1}$ be a braid in $\mathbf{B}_3$, where $\mathbf{B}_3$ is the braid group on 3 strings and σ_1, σ_2 are the standard Artin generators. We use Gauss diagram formulas to show that for each natural number n not divisible by 3 the knot which is represented by the closure of the braid β^n is algebraically slice if and only if n is odd. As a consequence, we deduce some properties of Lucas numbers.

1. Introduction

Let $\mathrm{Conc}(\mathbf{S}^3)$ denote the abelian group of concordance classes of knots in $\mathbf{S}^3$. Two knots $K_0, K_1 \in \mathbf{S}^3 = \partial\mathbf{B}^4$ are *concordant* if there exists a smooth embedding $c\colon \mathbf{S}^1 \times [0,1] \to \mathbf{B}^4$ such that $c(\mathbf{S}^1 \times \{0\}) = K_0$ and $c(\mathbf{S}^1 \times \{1\}) = K_1$. The knot is called *slice* if it is concordant to the unknot. The addition in $\mathrm{Conc}(\mathbf{S}^3)$ is defined by the connected sum of knots. The inverse of an element $[K] \in \mathrm{Conc}(\mathbf{S}^3)$ is represented by the knot $-K^*$, where $-K^*$ denotes the mirror image of the knot K with the reversed orientation.

Let $\mathrm{AConc}(\mathbf{S}^3)$ denote the algebraic concordance group of knots in $\mathbf{S}^3$. The elements of this group are equivalence classes of Seifert forms $[V_F]$ associated with an arbitrary chosen Seifert surface F of a given knot K. The addition in $\mathrm{AConc}(\mathbf{S}^3)$ is induced by direct sum. A knot K is called *algebraically slice* if it has a Seifert matrix which is metabolic. It is a well known fact that every slice knot is algebraically slice. For more information about these groups see [10].

Let $\mathbf{B}_3$ denote the Artin braid group on 3 strings and let σ_1, σ_2 be the standard Artin generators of $\mathbf{B}_3$, i.e. σ_i is represented by half-twist of $i+1$-th string over i-th string and $\mathbf{B}_3$ has the following presentation

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$$\mathbf{B}_3 = \langle \sigma_1, \sigma_2 | \, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle \, .$$

In this paper we discuss properties of a family of knots in which every knot is represented by a closure of the braid β^n, where $\beta = \sigma_1\sigma_2^{-1} \in \mathbf{B}_3$ and $n \neq 0 \bmod 3$. This family of braids is interesting in the following sense: the braid β has a minimal length among all non-trivial braids in $\mathbf{B}_3$ whose stable commutator length is zero. Hence by a theorem of Kedra and the author the four ball genus of every knot in this family is bounded by 4, see [2, Section 4.E.].

Theorem 1. *Let n be any natural number not divisible by 3. Then the closure of β^n is of order 2 in* $\mathrm{AConc}(\mathbf{S}^3)$ *if n is even and the closure of β^n is algebraically slice if n is odd.*

We would like to add the following remarks:

- The above theorem is not entirely new. The fact that the closure of β^n is a non-slice knot when n is even was proved by Lisca [9] using a celebrated theorem of Donaldson (also [14, Section 6.2] implies the same result). However, our proof of this fact is different. It uses Gauss diagram technique and is simple.
- The main ingredient of our proof is the computation of the Arf invariant. More precisely, we compute $\mathrm{Arf}(\widehat{\beta^n})$ for each n not divisible by 3. Note that if n is divisible by 3 then the closure of β^n is a three component totally proper link, and each of the components is a trivial knot. It follows from the result of Hoste [6] that its Arf invariant equals to the coefficient of z^4 of its Conway polynomial. In [1, Corollary 3.5] the author proved that this coefficient can be obtained as a certain count of ascending arrow diagrams with 4 arrows in a Gauss diagram of this link. However, in this case the computation is more involved since there are many ascending arrow diagrams with 4 arrows. It is left to an interested reader.
- It is still unknown whether the induced family of smooth or even algebraic concordance classes is infinite, and these seem to be hard questions.

Let $\{L_n\}_{n=1}^{\infty}$ be a sequence of Lucas numbers, i.e. it is a Fibonacci sequence with $L_1 = 1$ and $L_2 = 3$. Surprisingly, Theorem 1 has a corollary which is the following number theoretic statement.

Corollary 1. *Let $n \in \mathbf{N}$. Then*

(1) $L_{12n\pm4}$ *is equivalent to* $5 \bmod 8$ *or* $7 \bmod 8$*;*
(2) $L_{12n\pm2} \equiv 3 \bmod 8$*;*
(3) $L_{12n\pm2} - 2$ *is a square.*

Remark. Corollary 1 is not new. All parts of it can be proved directly. However, we think that it is interesting that a number theoretic result can be deduced from a purely topological statement. We would like to point out that the proof (identical to ours) of the fact that $L_{12n\pm2} - 2$ is a square for every n was given first in [14, Section 6.2].

2. Proofs

Let us recall the notion of a Gauss diagram.

Definition 2.1. Given a classical link diagram D, consider a collection of oriented circles parameterizing it. Unite two preimages of every crossing of D in a pair and connect them by an arrow, pointing from the overpassing preimage to the underpassing one. To each arrow we assign a sign (writhe) of the corresponding crossing. The result is called the *Gauss diagram G* corresponding to D.