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Entropy points of continuous maps with the sensitivity and the shadowing property

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ABSTRACT

We give some sufficient conditions that a point can be approximated by an entropy point in terms of the sensitivity and the shadowing property. More precisely, we prove that for a continuous self-map f of a compact metric space X and a closed f-invariant subset $S \subset X$, if eventually sensitive points of $f|_S$ are dense in S, then any point of S can be approximated by an entropy point with an accuracy corresponding to that of the shadowing. Moreover, its homeomorphisms version and two corollaries are proved.

1. Introduction

The sensitivity is one of the features of chaotic dynamical systems. Therefore, one may expect that such dynamical systems tend to have a positive topological entropy. However, as seen from an easy example given below, some additional condition is necessary to ensure that a system has a positive entropy besides the sensitivity. Recently, several results in this direction have appeared ([5–7]), which proved that systems of continuous maps with the sensitivity and the shadowing property have a positive topological entropy. In this paper, we provide sufficient conditions for the positive topological entropy in line with the above results.

We first give some basic definitions and notations. Let (X, d) be a compact metric space and let $f : X \to X$ be a continuous map. For $e > 0$, a point $x \in X$ is said to be an *e-sensitive point* of f if for any neighborhood U of x, there exist $y, z \in U$ and $n \in \mathbb{N}$ such that $d(f^n(y), f^n(z)) > e$. We denote by $\mathrm{Sen}_e(f)$ the set of

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e-sensitive points of f and define $\mathrm{Sen}(f) = \bigcup_{e>0} \mathrm{Sen}_e(f)$. A point of $\mathrm{Sen}(f)$ is called a *sensitive point* of f. We say that f is *sensitive* if $X = \mathrm{Sen}_e(f)$ for some $e > 0$, and such $e > 0$ is called a *sensitive constant* for f.

A finite or infinite sequence $(x_i)_{a\le i<b}$ of points in X, where $0 \le a < b \le \infty$, is a *δ-pseudo orbit* of f if $d(f(x_i), x_{i+1}) \le \delta$ for all $a \le i < b-1$. For given $\epsilon > 0$, a δ-pseudo orbit $(x_i)_{a\le i<b}$ is said to be *ϵ-shadowed* by $x \in X$ if $d(x_i, f^i(x)) \le \epsilon$ for all $a \le i < b$. For $S \subset X$ and $c > 0$, we say that f has a *c-shadowing property around* S if there is $\delta > 0$ such that every δ-pseudo orbit contained in S is c-shadowed by some $x \in X$. We say that f has the *shadowing property around* S if f has a c-shadowing property around S for every $c > 0$.

Given a continuous map $f : X \to X$ and $n \ge 1$, define a metric d_n on X by $d_n(x, y) = \max_{0\le j\le n-1} d(f^j(x), f^j(y))$. For $n \ge 1$ and $\epsilon > 0$, a subset $E \subset X$ is called *(n, ϵ)-separated* if $x \ne y$ $(x, y \in E)$ implies $d_n(x, y) > \epsilon$. For $A \subset X$, let $S(A, n, \epsilon)$ denote the maximal cardinality of an (n, ϵ)-separated set contained in A and consider

$$h(f, A, \epsilon) = \limsup_{n\to\infty} \frac{1}{n} \log S(A, n, \epsilon).$$

Note that $\epsilon_2 < \epsilon_1$ implies $h(f, A, \epsilon_1) \le h(f, A, \epsilon_2)$, which guarantees the existence of $\lim_{\epsilon\to 0} h(f, A, \epsilon)$. The *topological entropy of f on* A, denoted by $h(f, A)$, is $h(f, A) = \lim_{\epsilon\to 0} h(f, A, \epsilon)$. Then, the *topological entropy of* f, denoted by $h_{top}(f)$, is defined by $h_{top}(f) = h(f, X)$. In [9], Ye and Zhang introduced the notion of entropy points. A point $x \in X$ is said to be an *entropy point* of f if $h(f, \overline{U}) > 0$ for any neighborhood U of x. Let $\mathrm{Ent}(f)$ denote the set of entropy points of f. It is well known that $\mathrm{Ent}(f)$ is a closed f-invariant subset of X, and $h_{top}(f) > 0$ iff $\mathrm{Ent}(f) \ne \emptyset$ (see [8] and [9]).

Now let us state our first theorem.

Theorem 1.1. *Let $f : X \to X$ be a continuous map and let $S \subset X$ be a closed f-invariant subset. If there is $e > 0$ for which $\{x \in S : \omega(x) \cap \mathrm{Sen}_e(f|_S) \ne \emptyset\}$ is dense in S and f has a c-shadowing property around S with $0 < 2c < e$, then for any $x \in S$, there exists $y \in \mathrm{Ent}(f)$ such that $d(x, y) \le c$. In particular, if f has the shadowing property around S, then $S \subset \mathrm{Ent}(f)$.*

A point $x \in X$ is said to be a *recurrent point* of f if $x \in \omega(x)$, and the set of recurrent points of f is denoted by $\mathrm{R}(f)$. In [6], Moothathu proved that for a continuous map $f : X \to X$ with the shadowing property around a closed f-invariant subset $S \subset X$, letting $g = f|_S$, then for every $z \in S$, we have $z \in \mathrm{Ent}(f)$ if the following two conditions are satisfied:

(M1) $z \in \mathrm{Sen}(g)$;
(M2) $(z, z) \in \mathrm{Int}[\overline{\mathrm{R}(g \times g)}]$ (where the closure and the interior are taken in $S \times S$).

In Theorem 1.1, instead of the simultaneous recurrence condition (M2) for a certain sensitive point, we assume the density of eventually sensitive points and show that all points of S are entropy points especially when f has the shadowing property around S. Our result complements the results in [6].

When f is a homeomorphism, we define $\mathrm{Sen}^*_e(f) = \mathrm{Sen}_e(f) \cup \mathrm{Sen}_e(f^{-1})$. Then, we say that a homeomorphism f is *weakly sensitive* if $X = \mathrm{Sen}^*_e(f)$ for some $e > 0$, and such $e > 0$ is called a *weakly sensitive constant* for f. Here we should mention that some authors call such a homeomorphism a *sensitive* homeomorphism (see [1] for instance). The weak sensitivity can be understood as an extended notion of *expansiveness*. As for the relation between the positivity of topological entropy and the expansiveness, there is a classical result by Fathi [2], proving that if a homeomorphism $f : X \to X$ is expansive and $\dim X > 0$, then $h_{top}(f) > 0$. Later, Kato [4] generalized the result for continuum-wise expansive homeomorphisms. However, the weak sensitivity in itself does not necessarily yield a positive topological entropy as shown in the following example.