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Central Sets Theorem near zero

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In this paper, we introduce notions of J-set near zero and C-set near zero for a dense subsemigroup of $((0, +\infty), +)$ and state the Central Sets Theorem near zero. Among the other results for a dense subsemigroup $S \subseteq ((0, +\infty), +)$, we give some sufficient and equivalent algebraic conditions on a subset $A \subset S$ to be J-set near zero and to be C-set near zero.

1. Introduction

Let $(S, +)$ be a discrete semigroup. The collection of all ultrafilters on S is called the Stone–Čech compactification of S and denoted by βS. For $A \subseteq S$, define $\overline{A} = \{p \in \beta S : A \in p\}$, then $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets (also for the closed sets) of βS. We identify the principal ultrafilters with the points of S and thus pretend that $S \subseteq \beta S$. There is a unique extension of the operation to βS, making $(\beta S, +)$ a right topological semigroup (i.e. for each $p \in \beta S$, the right translation ρ_p is continuous, where $\rho_p(q) = q + p$) and also for each $x \in S$, the left translation λ_x is continuous, where $\lambda_x(q) = x + q$. The principal ultrafilters identified by the points of S and S is a dense subset of βS. For $p, q \in \beta S$ and $A \subseteq S$, we have $A \in p + q$ if and only if $\{x \in S : -x + A \in q\} \in p$, where $-x + A = \{y \in S : x + y \in A\}$.

A nonempty subset L of a semigroup $(S, +)$ is called a left ideal of S if $S + L \subseteq L$, a right ideal if $L + S \subseteq L$, and a two sided ideal (or simply an ideal) if it is both a left and a right ideal. A minimal left ideal is a left ideal that does not contain any proper left ideal. In the same way, we can define minimal right ideal and smallest ideal.

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Any compact Hausdorff right topological semigroup $(S,+)$ has a smallest two sided ideal, denoted by $K(S)$, which is the union of all minimal left ideals, and also the union of all minimal right ideals, as well. Given a minimal left ideal L and a minimal right ideal R, $L \cap R$ is a group and in particular contains an idempotent. An idempotent in $K(S)$ is called a minimal idempotent. For more details see [8].

For $A \subseteq S$ and $p \in \beta S$, we define $A^*(p) = \{s \in A : -s + A \in p\}$.

Lemma 1.1. *Let $(S,+)$ be a semigroup, $p + p = p \in \beta S$, and let $A \in p$. Then for each $s \in A^*(p)$, $-s + A^*(p) \in p$.*

Proof. See Lemma 4.14 in [8]. □

Now we review the definition of partition regularity. In this paper, the collection of all nonempty finite subsets of S is denoted by $P_f(S)$ and $\mathcal{P}(S)$ is the set of all subsets of S.

Definition 1.1. Let $\mathcal{R}$ be a nonempty set of subsets of S. $\mathcal{R}$ is partition regular if and only if whenever $\mathcal{F}$ is a finite subset of $\mathcal{P}(S)$ and $\bigcup \mathcal{F} \in \mathcal{R}$, there exist $A \in \mathcal{F}$ and $B \in \mathcal{R}$, such that $B \subseteq A$.

Theorem 1.2. *Let $\mathcal{R} \subseteq \mathcal{P}(S)$ be a nonempty set and assume $\emptyset \notin \mathcal{R}$. Let*

$$\mathcal{R}^{\uparrow} = \{B \in \mathcal{P}(S) : A \subseteq B \text{ for some } A \in \mathcal{R}\}.$$

Then (a), (b) *and* (c) *are equivalent.*

(a) *$\mathcal{R}$ is partition regular.*
(b) *Whenever $\mathcal{A} \subseteq \mathcal{P}(S)$ has the property that every finite nonempty subfamily of $\mathcal{A}$ has an intersection which is in $\mathcal{R}^{\uparrow}$, there is $\mathcal{U} \in \beta S$, such that $\mathcal{A} \subseteq \mathcal{U} \subseteq \mathcal{R}^{\uparrow}$.*
(c) *Whenever $A \in \mathcal{R}$, there is $\mathcal{U} \in \beta S$ such that $A \in \mathcal{U} \subseteq \mathcal{R}^{\uparrow}$.*

Proof. [8, Theorem 3.11]. □

Definition 1.3. Let $(S,.)$ be a discrete semigroup and $A \subseteq S$. Then A is a central set if and only if there exists an idempotent $p \in K(\beta S)$ with $A \in p$.

We have been considering semigroups which are dense in $((0,\infty),+)$ with the natural topology. When discussing the Stone–Čech compactification of such a semigroup S, we will deal with S_d, which is the set S with the discrete topology.

Definition 1.4. Let S be a dense subset of $((0,\infty),+)$. Then

$$0^+(S) = \{p \in \beta S_d : (\forall \epsilon > 0)\ (0,\epsilon) \cap S \in p\}.$$

By Lemma 2.5 in [7], $0^+(S)$ is a compact right topological subsemigroup of $(\beta S_d,+)$, and $0^+(S) \cap K(\beta S_d) = \emptyset$. Since $0^+(S)$ is a compact right topological semigroup, so $0^+(S)$ contains idempotents.

The set $0^+(S)$ of all non-principal ultrafilters on $S = ((0,\infty),+)$ that are convergent to 0 is a semigroup under the restriction of the usual '+' on βS_d, the Stone–Čech compactification of the discrete semigroup $S = ((0,\infty),+)$, see [7]. In [2], the authors used the algebraic structure of $0^+(S)$ in their investigation of image partition regularity near 0 of finite and infinite matrices.

In [5], the algebraic structure of $0^+(\mathbb{R})$ was used to investigate image partition regularity of matrices with real entries from $\mathbb{R}$. Central sets near zero were introduced by N. Hindman and I. Leader in [7] as