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Measures and fibers

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A B S T R A C T

We study measures on compact spaces by analyzing the properties of fibers of continuous mappings into 2^ω. We show that if a compact zerodimensional space K carries a measure of uncountable Maharam type, then such a mapping has a non-scattered fiber and, if we assume additionally a weak version of Martin's Axiom, such a mapping has a fiber carrying a measure of uncountable Maharam type. Also, we prove that every compact zerodimensional space which supports a strictly positive measure and which can be mapped into 2^ω by a finite-to-one function is separable.

1. Introduction

We say that a compact space K is *φ-fibered* (or *has fibers satisfying* φ) if there is a compact metric space M and a continuous function $f\colon K \to M$ such that $f^{-1}[\{x\}]$ has property φ for each $x \in M$. So, we can consider e.g. n-fibered, finitely-fibered, metrizably-fibered or spaces with scattered fibers.

Questions about properties of fibers of continuous mappings to metric spaces appear quite naturally in many contexts. The question if it is consistent that all perfectly normal compact spaces are 2-fibered is one of the most important questions of set theoretic topology (see [14]). Metrizably- and finitely-fibered spaces were considered in the context of spaces with a small diagonal (see e.g. [15,7]) and of Rosenthal compacta (see e.g. [16]). Non-separable linearly-fibered spaces are in a sense direct generalizations of the Suslin line and were studied by Moore [20] and Todorčević ([26], see also Example 4.4).

The systematic study of metrizably-fibered spaces was undertaken in [25]. Independently, Tkachenko [24] considered a slightly weaker notion than being φ-fibered. We say that a space K is *φ-approximable*

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if there is a countable cover by closed sets such that the maximal intersections of elements of this cover have property φ (by replacing "closed sets" with "zero sets" we obtain an equivalent definition of φ-fibered space, see [25, Proposition 2.1]). Tkachenko investigated e.g. for which properties φ spaces satisfying φ are φ-approximable.

In this article we will examine how the existence of certain types of measures affects the properties of fibers. For the sake of this section say that a compact space K has property $A(\kappa)$ if it can be mapped continuously onto $[0,1]^\kappa$ and K has property $M(\kappa)$ if it carries a measure of Maharam type κ (see Section 2 for the definitions). Loosely speaking by determining for which κ a compact space has $A(\kappa)$ we can measure its *combinatorial complexity* and similarly using $M(\kappa)$ we can measure its *measure-theoretic complexity*.

Note that for each κ the property $A(\kappa)$ implies $M(\kappa)$. In general, the converse holds only for some κ's. For example, the scattered spaces (i.e. those which do not have $A(\omega)$) only carry purely atomic measures and so they do not have $M(\omega)$. Fremlin proved that under MA_{ω_1} the property $A(\omega_1)$ is equivalent to $M(\omega_1)$ and there are many consistent examples of spaces with $M(\omega_1)$ and without $A(\omega_1)$. For the results concerning the relationship between $A(\kappa)$ and $M(\kappa)$ we refer the reader e.g. to [22]. In this article we will be interested only in cases when $\kappa \leq \omega_1$.

Let K be a compact zerodimensional space and let $f\colon K \to 2^\omega$ be a continuous mapping. Of course, $A(\omega)$ is not necessarily inherited by fibers of f, since K having $A(\omega)$ can be even 1-fibered. However, Tkachenko proved that if K has $A(\omega_1)$, then one of the fibers of f has $A(\omega_1)$ (see [24], we reprove this result in Section 2).

The property $M(\omega_1)$ in this context behaves in a more complicated way. If $\mathrm{cov}(\mathcal{N}) = \omega_1$, then there is a compact space with $M(\omega_1)$ which has fibers homeomorphic to 2^ω, and so none of them has $M(\omega_1)$ (see Example 4.2). On the other hand, the theorems of Fremlin and of Tkachenko mentioned above imply that under MA_{ω_1} the property $M(\omega_1)$ has to be inherited by some fiber. In Section 2 we prove that under MA_{ω_1} for measure algebras (MA_{ω_1}(ma), in short) every zerodimensional space with $M(\omega_1)$ has a fiber with $M(\omega_1)$. Note that MA_{ω_1}(ma) is considerably weaker than MA_{ω_1}. In particular it is consistent with the existence of space with $M(\omega_1)$ and without $A(\omega_1)$. On the other hand, MA_{ω_1}(ma) is equivalent to $\mathrm{cov}(\mathcal{N}_{\omega_1}) > \omega_1$, which is just a slightly stronger axiom than $\mathrm{cov}(\mathcal{N}) > \omega_1$.

In Section 3 we show in ZFC that the zerodimensional spaces with $M(\omega_1)$ cannot have too *simple* fibers. Namely, for every continuous $f\colon K \to 2^\omega$, where K has $M(\omega_1)$, at least one fiber of f has $M(\omega)$. There are many properties of compact spaces implying that a space only carries separable measures (e.g. linearity, being a Rosenthal compactum, being a Stone space of a minimally generated Boolean algebra). Theorem 3.1 enlarges this list adding the property of being a space with scattered fibers. In Section 4 we provide some known examples of spaces with scattered fibers. Theorem 3.1 implies that all of them only carry separable measures.

Section 3 contains one more result of this sort. Say that a compact space has property $(\star)$ if it is non-separable and supports a (strictly positive) measure. In a sense such spaces are big from the measure-theoretic point of view, similarly to spaces having $M(\omega_1)$. There are spaces having $M(\omega_1)$ but not $(\star)$ (e.g. 2^{ω_1}). The question if $(\star)$ implies $M(\omega_1)$ is quite interesting and, in fact, it was one of the motivations for our research (see [6] for further discussion). It turns out that there are, at least consistently, non-separable zerodimensional spaces supporting measures and having scattered fibers (see remarks at the end of Section 4). In light of Theorem 3.1 those spaces have $(\star)$ but not $M(\omega_1)$. In Section 3 we prove Theorem 3.4 saying that the spaces with $(\star)$ at least cannot be finitely-fibered.

In Section 4 we present some relevant examples and we give some additional motivation for our research.

Let us mention that although the authors mentioned at the beginning of this section have not considered properties connected to measures in the context of fibers, the study of measures on fibers of measurable mappings is an important part of probability theory. It is worth to recall here well-known Rokhlin's theorem: