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Gruff ultrafilters ☆

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ABSTRACT

We investigate the question of whether $\mathbb{Q}$ carries an ultrafilter generated by perfect sets (such ultrafilters were called *gruff ultrafilters* by van Douwen). We prove that one can (consistently) obtain an affirmative answer to this question in three different ways: by assuming a certain parametrized diamond principle, from the cardinal invariant equality $\mathfrak{d} = \mathfrak{c}$, and in the Random real model.

1. Introduction

In a 1992 paper, Eric van Douwen [7] carried out an investigation about certain points in the Čech–Stone compactification of $\mathbb{Q}$ (where $\mathbb{Q}$ is equipped with the topology inherited from the Euclidean topology on $\mathbb{R}$, so that points in $\beta\mathbb{Q}$ can be realized as maximal filters of closed sets), with the property that they actually generate an ultrafilter on $\mathbb{Q}$. In other words, van Douwen was looking at ultrafilters over $\mathbb{Q}$ that have a base

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of closed sets, and among those he paid particular attention to the ones where the elements of a base can be taken to be crowded (recall that a set is crowded if it has no isolated points), in addition to being closed. This was the motivation for stating the following definition.

Definition 1.1. A nonprincipal ultrafilter u on $\mathbb{Q}$ is said to be **gruff** (a pun on the fact that these are points in $\beta\mathbb{Q}$ that "generate real ultra filters") if it has a base of perfect (i.e. closed and crowded) subsets of $\mathbb{Q}$. This is, we require that $(\forall A \in u)(\exists X \in u)(X \text{ is perfect and } X \subseteq A)$.

Recall that a coideal on a set X is a family $\mathscr{A}$ with the property that $\varnothing \notin \mathscr{A}$, $\mathscr{A}$ is closed under supersets, and whenever an element $A \in \mathscr{A}$ is written as $A = A_0 \cup A_1$, there exists an $i \in 2$ such that $A_i \in \mathscr{A}$. If the infinite set X has a topology in which X itself is crowded, then the family

$$\mathscr{C} = \{A \subseteq X \big| A \text{ contains an infinite crowded set}\}$$

constitutes a coideal. Moreover, in the topological space $\mathbb{Q}$, every infinite crowded set contains an infinite perfect subset. This fact, which is not true in a general topological space (for example, in every Polish space it is possible to construct *Bernstein sets*, sets that are not contained in nor disjoint from any uncountable perfect subset), implies that the family

$$\mathscr{P} = \{A \subseteq X \big| A \text{ contains an infinite perfect set}\},$$

also constitutes a coideal on $\mathbb{Q}$. It is for this reason that Definition 1.1 is justified.

The main question that van Douwen asked about gruff ultrafilters is whether their existence can be proved in ZFC. He himself [7, Thm. 2.1] provided a partial answer by proving that the existence of a gruff ultrafilter follows from the cardinal invariant equality $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$, which is equivalent to Martin's Axiom restricted to countable forcing notions. Although the question remains open, more partial results have been proven. Copláková and Hart [6, Thm. 1] proved in 1999 that the existence of a gruff ultrafilter follows from $\mathfrak{b} = \mathfrak{c}$. Some time after, in 2003, Ciesielski and Pawlikowski [3, Thm. 4.22] showed that the existence of a gruff ultrafilter follows from a combinatorial principle known as $\mathsf{CPA}^{\text{game}}_{\text{prism}}$, which, in particular, implies that there exist gruff ultrafilters in the Sacks model, as this model satisfies that combinatorial principle. This theorem was improved shortly after by Millán [8, Thm. 3], who showed that, in fact, $\mathsf{CPA}^{\text{game}}_{\text{prism}}$ implies the existence of a gruff ultrafilter that is at the same time a Q-point (it is shown in [4, Prop. 5.5.5] that a gruff ultrafilter cannot be a P-point).

In this paper, we obtain three more partial answers to van Douwen's question. The first result involves the theory of diamond principles that are parametrized by a cardinal invariant, as developed in [9]. We define a cardinal characteristic $\mathfrak{r}_P$ that relates naturally to perfect subsets of $\mathbb{Q}$, and show that its corresponding parametrized diamond principle $\diamondsuit(\mathfrak{r}_P)$ implies the existence of a gruff ultrafilter which is at the same time a Q-point. We also show that this parametrized diamond principle holds in the Sacks model, thus providing an alternative proof of Millán's theorem on the existence of gruff Q-points in this model. Our second result is that the existence of gruff ultrafilters follows from the cardinal invariant equality $\mathfrak{d} = \mathfrak{c}$. Since (it is provable in ZFC that) $\mathfrak{d} \geq \mathfrak{b}$ and $\mathfrak{d} \geq \mathrm{cov}(\mathcal{M})$, but both inequalities can be consistently strict (even simultaneously), this result is stronger than both van Douwen's, and Copláková and Hart's. Finally, our third result is that in the Random real model there exists a gruff ultrafilter (this, together with the $\diamondsuit(\mathfrak{r}_P)$ result mentioned above, shows that the existence of gruff ultrafilters is consistent with $\mathfrak{d} < \mathfrak{c}$). First we will prove a lemma that will simplify our interaction with gruff ultrafilters. Throughout this paper, the notation (a, b) will be used to refer to intervals on $\mathbb{Q}$. In other words, $(a, b) = \{q \in \mathbb{Q} \big| a < q < b\}$ whenever $a, b \in \mathbb{R}$.

Lemma 1.2. *There exists a gruff ultrafilter on $\mathbb{Q}$ if and only if there exists an ultrafilter on the set of positive rational numbers $\mathbb{Q}^+$ with a base of perfect unbounded sets.*